Differential Equations - Notes

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Office Hours:
Please remember to check the class website for office hours, homework assignments, and other helpful information.

Ordinary Differential Equations - Day 6

Today we will talk about a solution method that works with special cases of nonlinear ordinary differential equations. Last time we talked about substitution methods, but what if neither of the substitution methods is working for you. Your next best bet is to see if the equation is in exact form.

Exact Equations

Our goal, much like with linear first order equations, is to reduce the problem of solving a differential equation to the problem of a much simpler integral. Here we are focusing of equations of the form:

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0$$

where M and N satisfy the condition

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Derivation of the Method - Exact Equations

Let's take a look at this from a goal oriented point of view! Say you are given an ODE that you can't solve using any of the methods we have so far.

$$\frac{dy}{dx} = f(x,y) = \frac{-M(x,y)}{N(x,y)}$$

here we are just breaking up our $f(x,y)$ into two functions. Then say you bring everything to the LHS:

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{1}$$

I will call this form of the ODE Eqn 1. Wouldn't it be great if we could rewrite the LHS so that it is the derivative of some function? In other words

$$M(x,y) + N(x,y)\frac{dy}{dx} = \frac{d}{dx}[F(x,y)] = 0$$

If we could make this magic happen, aka find this mysterious function $F(x,y)$, then our problem is done and our solution is

$$F(x,y) = c$$

So, let's investigate this idea! What conditions would we need to impose on M and N to make sure that F exists? How do we find F if it does exist? We will answer these questions by working backward. I start with what I want, and see what mathematics has to say about that...

Okay, I want to rewrite my ODE in the form:

$$\frac{d}{dx}[F(x,y)] = 0$$

We can easily take this derivative on the LHS, remembering that y is a function of x and thus we will need to use the chain rule.

$$\frac{d}{dx}[F(x,y)] = \frac{\partial F}{\partial x} + \frac{\partial F}{\partial y}\frac{dy}{dx} = 0$$

Wait a second! This looks a heck of a lot like what I wrote as my exact form in Eqn 1. So this is math telling me that I can define F in terms of M and N.

$$\frac{\partial F}{\partial x} = M(x,y) \qquad \frac{\partial F}{\partial y} = N(x,y)$$

This means that you can just integrate M and N to find this wonderful function F. But, before we go out celebrating, how do we know that these two different integrals will lead to the same F? In other words, how do we know that this single function F exists? It actually comes back to Calc III if we take the partial derivative of M with respect to y and the partial derivative of N with respect to x we get

$$\frac{\partial^2 F}{\partial x \partial y} = \frac{\partial M(x,y)}{\partial y} \qquad \frac{\partial^2 F}{\partial y \partial x} = \frac{\partial N(x,y)}{\partial x}$$

and what do we know about mixed partial derivatives of F? They must be equal! So if the function F exists and can be defined by M and N then we can just check that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

And that's it, this gives us our method for solving exact equations.

Exact Method

Here is the way that we use the method in practice

1. First we write the equation in exact form. Or recognize that it is already in exact form.
2. Then we test to see if our function F exists. In other words we check that
$$\frac{\partial M(x,y)}{\partial y} = \frac{\partial N(x,y)}{\partial x}$$
3. If the partial derivatives match then we can define F as BOTH
$$\frac{\partial F}{\partial x} = M(x,y) \qquad \frac{\partial F}{\partial y} = N(x,y)$$
4. We should be able to compare these two results and find a single function $F(x,y)$ that works and our solution is given by
$$F(x,y) = c$$

Let's look at an example

EXAMPLE:

$$y^3 + 3xy^2\frac{dy}{dx} = 0$$

First, sometimes the book and others will write this as

$$y^3\ dx + 3xy^2\ dy = 0$$

It means the same thing so don't freak out!

1. First we need to check that the equation is in exact form. Yes it is and here we have $M(x,y) = y^3$ and $N(x,y) = 3xy^2$.
2. Next we need to check that $F(x,y)$ exists by taking the partial derivatives
$$\frac{\partial M}{\partial y} = 3y^2 \text{ and } \frac{\partial N}{\partial x} = 3y^2$$
so our exact solution method should result in a single function F that works!
3. Next we solve for F twice! First using the M equation
$$\frac{\partial F}{\partial x} = M(x,y) = y^3$$
Integrate this with respect to x
$$F(x,y) = xy^3 + g(y)$$
notice here that because I am dealing with a function of two variables the "constant" of integration is actually any function of y. Check this... if you take the partial derivative of that function you get back M. Now we will solve for F using the N equation
$$\frac{\partial F}{\partial y} = N(x,y) = 3y^2$$
integrate this with respect to y
$$F(x,y) = xy^3 + h(x)$$
again our "constant" of integration is any function of x.
4. Now we compare these two results and balance them both together.
$$xy^3 + h(x) = xy^3 + g(y) = F(x,y)$$
so we can choose $h(x) = g(y) =$ any constant and here we will pick zero. Which means $F(x,y) = xy^3$ and the solution to our ODE is
$$xy^3 = c$$
This is our general solution!

How would we check that it works? Take the derivative of both sides and you should get back your original ODE.
$$\frac{d}{dx}[xy^3] = y^3 + 3xy^2\frac{dy}{dx} \qquad \frac{d}{dx}[c] = 0$$
which gives
$$y^3 + 3xy^2\frac{dy}{dx} = 0$$
Be careful here to use the product rule on the LHS!

EXAMPLE:

$$(6xy - y^3) + (4y + 3x^2 - 3xy^2)\frac{dy}{dx} = 0$$

This one will be slightly more challenging but it uses the same basic steps.

1. The equation is already given to us in exact form.
2. Test that F exists, aka that the equation is really exact.
$$\frac{\partial F}{\partial y} = 6x - 3y^2$$
$$\frac{\partial F}{\partial x} = 6x - 3y^2$$
These are the same so there is some function F that will solve our equation using the exact method!

3. Now find $F(x,y)$ by integration

$$F(x,y) = \int M(x,y)\, dx = \int 6xy - y^3\, dx$$
$$= 3xy^2 - xy^3 + g(y)$$

$$F(x,y) = \int N(x,y)\, dy = \int 4y + 3x^2 - 3xy^2\, dy$$
$$= 2y^2 + 3x^2y - xy^3 + h(x)$$

4. Now compare these two results

$$3x^2y - xy^3 + g(y) = 2y^2 + 3x^2y - xy^3 + h(x)$$

Here we see that we must define $h(x) = 0$ and $g(y) = 2x^2$ to make both sides of the equation the same so our solution is

$$F(x,y) = 2y^2 + 3x^2y - xy^3 = c$$

YOU TRY:

$$(2x+3y) + (3x+2y)\frac{dy}{dx} = 0$$

Answer [1]

$$(6y-x)\frac{dy}{dx} - y = -4x$$

Answer [2]

Reducible Second Order Equations

So far we have learned lots of methods for solving first order equations and only one method for higher order equations. Remember that we can solve

$$\frac{d^ny}{dx^n} = f(x)$$

by integrating n times. But what about all the other higher order equations that are out there? Don't we live in a world beyond velocity!?!

There are two very straightforward cases of second order equations that can easily be transformed into first order equations.

1. Equations that do not have the dependent variable showing explicitly.

2. Equations that do not have the independent variable showing explicitly.

[1] $x^2 + y^2 + 3xy = c$
[2] $2x^2 - xy + 3y^2 = c$
[3] $y(x) = x^2 + \frac{c_1}{x} + c_2$

We will consider each of these cases independently.

1. Missing the dependent variable:

If we are missing the dependent variable then we can use the substitution

$$u = \frac{dy}{dx} \quad \text{and} \quad u' = \frac{d^2y}{dx^2}$$

After making the substitution you should be able to solve the new ODE for u with methods we already know and then integrate once to get back to y.

EXAMPLE:

$$\frac{d^2y}{dx^2} + \frac{dy}{dx} = x$$

Here we see a second order ODE that does not have y explicitly in it. We ONLY see derivatives of the dependent variable not the variable itself. So the substitution

$$u = \frac{dy}{dx} \quad \text{and} \quad u' = \frac{d^2y}{dx^2}$$

will work. Plugging in we get

$$u' + u = x$$

Now this is a first order linear equation that we can solve with an integrating factor.

$$\begin{aligned}
\rho(x) &= e^{\int dx} = e^x \\
e^x u' + e^x u &= e^x x \\
\frac{d}{dx}\left[e^x u\right] &= xe^x \\
e^x u = \int xe^x\, dx &= xe^x - e^x + c_1 \\
u &= x - 1 + c_1 e^{-x}
\end{aligned}$$

Now we need to integrate to get our solution y

$$\begin{aligned}
u = \frac{dy}{dx} &= x - 1 + c_1 e^{-x} \\
y &= \int x - 1 + c_1 e^{-x}\, dx \\
y(x) &= \frac{1}{2}x^2 - x - c_1 e^{-x} + c_2
\end{aligned}$$

YOU TRY:

$$xy'' + 2y' = 6x$$

Answer [3]

2. `Missing the independent variable:`

If we are missing the independent variable then we use the substitution

$$p = \frac{dy}{dx} = y' \text{ and } y'' = \frac{d}{dx}[p] = \frac{dp}{dy}\frac{dy}{dx} = p\frac{dp}{dy}$$

Notice here that p is a function of y and y is a function of x, so in the second expression we needed to use the chain rule to take the derivative with respect to x. After making the substitution you should have a first order equation that you can solve with existing methods. Once you solve for p you can solve one more first order ODE to get back to y.

EXAMPLE:

$$yy'' = (y')^2$$

This is a second order, very nonlinear problem. Here we see that the independent variable x does not appear explicitly in the equation so we know that we can use the above substitution to reduce it to a first order equation. We let

$$p = \frac{dy}{dx} = y' \text{ and } y'' = p\frac{dp}{dy}$$

Subbing into the ODE

$$yp\frac{dp}{dy} = p^2 \rightarrow \frac{dp}{dy} = \frac{p}{y}$$

This is a first order separable ODE that we can solve for the new variable p

$$\frac{dp}{dy} = \frac{p}{y}$$
$$\frac{dp}{p} = \frac{dy}{y} \rightarrow \int \frac{dp}{p} = \int \frac{dy}{y}$$
$$\ln|p| = \ln|y| + c_1 \text{ so } p = Ay$$

Here we used the trick of pulling a constant out of the exponent. Now we can solve for y using separation:

$$\frac{dy}{dx} = p = Ay$$
$$\frac{dy}{y} = A\,dx \rightarrow \int \frac{dy}{y} = \int A\,dx$$
$$\ln|y| = Ax + B \rightarrow y = e^{Ax+B} = Ce^{Ax}$$

YOU TRY:

$$y^3y'' = 1$$

This one is a bit tricky and it helps to solve the substitution

$$\frac{dy}{dx} = p$$

for x. Since it is separable we can write

$$\int \frac{dy}{p} = \int dx = x$$

Once we find p we can sub it back into

$$x = \int \frac{1}{p}\,dy$$

do the integral and then simplify to try to solve for y. Answer [4]

[4] $y^2 = \frac{1}{A}\left[(Ax + C)^2 + 1\right]$