Differential Equations - Notes

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Office Hours:
Please remember to check the class website for office hours, homework assignments, and other helpful information.

Ordinary Differential Equations - Day 7

Now that we have many different solution methods, we will investigate some applications of First order ODEs. This is just a small sample of the many different systems that can be modeled with first order equations.

Mathematical Models

Even though we are focusing on populations here, note that this idea or method could be applied much more broadly. For example the process we are about to use could be used to model money in a bank account, bacteria in a sample, or even the increase or decrease of emotions. Just keep an imaginative mind!

The general model building process:

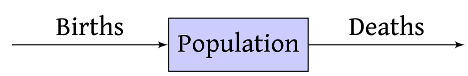
1. What do we want to know?
   - What is the thing that is changing over time?
   - Define the units. (time and population numbers)
2. What do we know about the system?
   - Define variables
   - Make assumptions
   - Find rates of change and initial conditions
3. Formulate the ODE
   - Draw a compartmental diagram
   - Write down the equations
4. Solve the ODE
   - Classify and choose a solution method
   - Investigate the solution. Does it make sense? What is it telling you about the population?

Population Models

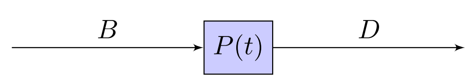
When modeling populations, what are the most most fundamental things that change the population over time? That's right! Births and Deaths.So we are going to start simple and imagine a closed system with no explicit environmental effects. In other words, the population does not migrate and our births and deaths are constants.

- Constant number of births B per unit time.
- Constant number of deaths D per unit time.

Now we want to draw a compartmental diagram:



Here you should see a box that contains the dependent variable, population. You always put the variable that you are tracking, aka the variable that is changing, inside the box. Then you should see arrows going in with anything that increases the quantity and arrows flowing out with anything that decreases the quantity. Here we see births flowing in and deaths flowing out. Now adding symbols:



This allows us to easily write down a first order equation. The box represents a rate of change, for us that means derivative the in and out flows should be added and subtracted respectively:

$$\frac{dP}{dt} = B - D$$

Now we can refine this model to be more realistic by taking a closer look at B and D.

Population Models - Constant Rates

Usually when modeling a population over time you can say something about the number of offspring per animal per time step or the rate of deaths per animal per time step. Think about this in terms of humans we can say there are 12.5 births per month for every 1000 humans so $b = 0.0125$ with units of $\frac{1}{months}$. Generalizing this is as simple as defining

$$B = bP(t)$$

where b is a constant that represents how the population is magnified per unit time by births. The units of b should be $\frac{1}{time}$, so this is similar to a frequency in phyiscs. b is called a birth rate, or in some cases a fecundity. We can make the same assumption about deaths

$$D = dP(t)$$

This allows us to write our equation as

$$\frac{dP}{dt} = bP - dP = (b-d)P = rP$$

where $r = (b-d)$ is an overall growth rate.

We can expand these models quite easily. For example what if we added a constant hunting term?

$$\xrightarrow{\;bP(t)\;} \boxed{P(t)} \xrightarrow{\;dP(t)+H\;}$$

In other words each time step H members of the population are removed. Then our ODE becomes:

$$\frac{dP}{dt} = rP - H$$

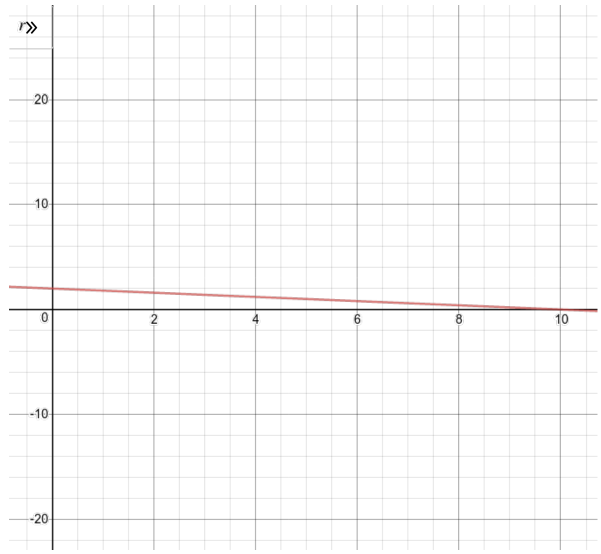
where H is some constant, a number. Notice that in all of these cases we are left with a separable ODE that we can solve!

Population Models - Non-Constant Rates

Often we will want to define the birth, death, or growth rates as a function of population. In our above model this is like saying $r = f(P)$, or the growth rate depends on the value of the population. For example, in a closed system with limited but renewable resources, the amount of growth of a population depends on how many members of the population are using the resource. In simple words, if too many animals are eating up all the food then the growth rate decreases and even possibly goes negative until the population is low enough to balance with the growth of food.

- When there are too many animals at some point the growth rate r goes to zero.
- When there are very few animals the growth rate r would reach some maximum R_{max}.
- We need to assume what the function $r = f(P)$ does in between it's max and zero.

One very common assumption is that $r = f(P)$ is a linear function. So if we imagine a straight line on a graph of r vs P it would look like



Want to make pretty graphs? I use desmos.com a free online graphing tool to make this graph. It even has sliders so you can play with parameters and you can easily enter more than one equation

Here we will define two important points:

- The y-intercept is the maximum growth rate R, imagine this as the growth rate as the population gets very close to zero.
- the x-intercept is the zero growth rate, when the population reaches the resource limit. We will call this K, the environmental carrying capacity.

We can write down the equation for this line.

$$r = \frac{-R}{K}P + R = R\left(1 - \frac{P}{K}\right)$$

This new functional growth rate can be plugged into our very basic population growth without hunting ODE

$$\frac{dP}{dt} = rP = R\left(1 - \frac{P}{K}\right)P$$

The Moral(s) of the Story

- It is actually quite easy to draw compartmental diagrams that represent real world problems. Try it!! You could model diet, calories in calories out, or your bank account, money in money out.
- after you have a compartmental diagram writing down the ODE is fairly simple.
- This modeling method will only work for first order equations and you need to be careful of how you are defining your units. Everything must actually be a continuous function or parameter.