Differential Equations - Notes

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Office Hours:
Please remember to check the class website for office hours, homework assignments, and other helpful information.

Ordinary Differential Equations - Day 7

We will continue talking about applied problems.

Mixture Problems

Mixture problems are problems that consist of a mixture of two things. Usually we think of a solute and solvent, or two fluids perfectly mixed. In these problems we are thinking about some amount flowing into a system and some amount flowing out.

EXAMPLE:
You got sugar in my coffee!!

Here is the situation. Ross (my husband) and I have different tolerance for sugar and milk in our coffee. I like only a little bit, 30% sugar milk concentrate by volume, and Ross likes a lot, 50% sugar milk concentrate by volume, Ross makes coffee each morning and mixes the sugar/milk concentrate directly into the coffee pot. There is no way that we have time to mix up our own coffee so instead I come up with a contraption!

I start with 1 gallon of pure coffee, and then have machine pour in sugar milk concentrate at exactly .2 gallons per hour. Out of the other end flows the perfectly mixed coffee at a rate of .3 gallons per hour. So now that I have this contraption sitting on my living room coffee table, my question to the class is: When should I grab my cup of coffee and when should Ross grab his?

We will solve this as a mixture problem. Lets define $x(t)$ as the amount of sugar milk concentrate in the coffee mix. So at $t = 0$ $x(0) = 0$ meaning I have only pure black coffee. Okay now the change in the amount of milk sugar concentrate in our coffee contraption depends only on the amount flowing in and the amount flowing out:

$$\frac{dx}{dt} = \text{amt. flowing in} - \text{amt. flowing out}$$

How much is flowing in? Well we said above that this is .2 gallons per hour. So we can plug that number in

$$\frac{dx}{dt} = 0.2 - \text{amt. flowing out}$$

Now how much is flowing out? Well we know that .3 gallons per hour of the mixture leave the pot, but how much of the sugar milk concentrate leave the pot? This is harder to figure out! Lets think about units, flowing out is

$$0.3 \frac{\text{gallons of mix}}{\text{hour}}$$

But wait, we want to know how much concentrate (not mix) is flowing out! So we need to multiply by

$$\frac{\text{gallons of concentrate}}{\text{gallons of mix}}$$

to get the correct units. This leads to the question: At time t how much mixture is in the tank and how much concentrate is in the tank? At time t the amount of concentrate is given by $x(t)$. Then if $.2$ gallons of concentrate is flowing in and $.3$ of the mix is flowing out, then we would have $(1 - .1t)$ mix at time t, the coffee tank is slowly draining. Thus

$$\frac{\text{gallons of concentrate}}{\text{gallons of mix}} = \frac{x(t)}{(1 - 0.1t)}$$

So the amount of mixture leaving the pot is

$$\frac{0.3x}{(1 - 0.1t)} \frac{\text{gallons of mix}}{\text{hour}}$$

So our ode becomes

$$\frac{dx}{dt} = .2 - \frac{.3x}{(1 - .1t)}, \quad x(0) = 0$$

This is a linear first order non-separable ODE and we can solve it using an integrating factor!

$$\rho(x) = e^{\frac{.3}{1-.1t}} = (1 - .1t)^{-3}$$

multiply through by $\rho(x)$ and recognize the derivative of a product

$$\frac{d}{dx}((1 - .1t)^{-1}x) = .2(1 - .1t)^{-3}$$

integrating we find

$$x = (1 - /1y) + C(1 - .1t)^3$$

as our general solution. Then applying the initial condition

$$x(0) = 0 = 1 + C \text{ so } C = -1$$

and

$$x(t) = (1 - .1t) - (1 - .1t)^3$$

Now we need to answer the original question! When should I get my coffee and when should Ross get his? I like to have 10% milk sugar concentrate so I need to figure out this percentage!

$$\frac{x(t)}{v(t)}$$

where $v(t)$ is the volume of mixture at time t and $x(t)$ is the amount of concentrate.

$$\frac{x(t)}{v(t)} = \frac{(1 - .1t) - (1 - .1t)^3}{(1 - .1t)} = 1 - (1 - .1t)^2$$

Now I want

$$.3 = 1 - (1 - .1t)^2$$

and solving for t I get $t = 1.633$ hours. Notice here that I would need to choose the correct one of the solutions to the quadratic formula. But, I know that t is not equal to 18.366 because my coffee pot is empty after ten hours. Similarly I can solve for Ross

$$.5 = 1 - (1 - .1t)^2$$

and I find that Ross should grab his coffee at $t = 2.9289$ hours. Assuming I don't increase the flow rate and drink all the coffee before he gets some, mwa hahaha!

The Moral(s) of the Story

- Drawing compartmental diagrams helps to simplify your thinking about a problem.
- Units are especially helpful in figuring out what some of these expressions should be.