Lecture 23 Diff Eq

4.1 First Order Systems – Lots of reasons to study first order systems!

1. Many systems have more than one variable or component associated with them.

<u>Ex</u> A particle moving through 3-D space:

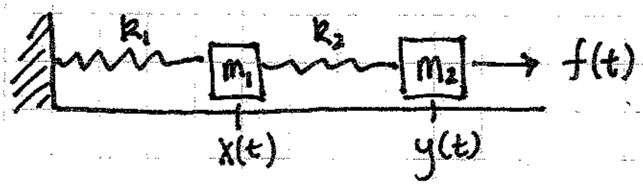
$\vec{F} = m\vec{a}$ this is a VECTOR ODE!

$$\vec{F} = m\frac{d^2\vec{x}}{dt^2} \qquad \vec{F} = \langle F_1, F_2, F_3 \rangle \qquad \vec{x} = \langle x, y, z \rangle$$

$$\begin{bmatrix} F_1 \\ F_2 \\ F_3 \end{bmatrix} = m\frac{d^2}{dt^2}\begin{bmatrix} x \\ y \\ z \end{bmatrix} \Rightarrow \begin{aligned} m\ddot{x} &= F_1 \\ m\ddot{y} &= F_2 \\ m\ddot{z} &= F_3 \end{aligned} \rightarrow \text{a system of equations for 3-D motion.}$$

↳ second order.

<u>Ex</u> Two masses and two springs: Coupled Oscillators.



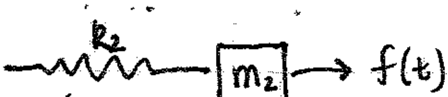
$x(t)$ – displacement from equilibrium mass one

$y(t)$ – displacement from equilibrium mass two.

spring force $-k_1 x$

force from stretching or compressing spring two $k_2(y-x)$

↳ depends on y-location.



force from spring $-k_2(y-x)$ equal and opposite push.

$\Sigma F = ma$ in each case

$$m_1 \ddot{x} = -k_1 x + k_2(y-x)$$
$$m_2 \ddot{y} = -k_2(y-x) + f(t)$$

This is a linear system of second order ODE's.

2. Any higher order equation can be reduced to an equivalant system of first order eqns.

- most numerical methods for ode's are developed only to first order.
- in non linear dynamics we focus only on first order!

An n^{th} order equation can be written as

$$x^{(n)} = f(t, x, x', x'', \ldots, x^{(n-1)})$$

solve for highest derivative on LHS

we can introduce variables

$$\begin{aligned} x_1 &= x \\ x_2 &= x' = x_1' \\ x_3 &= x'' = x_2' \\ &\vdots \\ x_n &= x^{(n-1)} = x_{n-1}' \end{aligned} \quad \Rightarrow \quad \begin{aligned} x_1' &= x_2 \\ x_2' &= x_3 \\ &\vdots \\ x_{n-1}' &= x_n \\ x_n' &= f(t, x_1, x_2, \ldots, x_{n-1}) \end{aligned}$$

This is our first order system.

<u>EX</u> $x^{(3)} + 3x'' + 2x' - 5x = \sin 2t$

1. Solve for the highest derivative

$$x^{(3)} = \sin 2t - 3x'' \mp 2x' + 5x$$

2. Introduce enough variables to cover highest derivative... on RHS.

$$x_1 = x$$
$$x_2 = x'$$
$$x_3 = x''$$

3. Rewrite these so first derivative appears on LHS of each

$$\begin{aligned} x_1 &= x \\ x_2 &= x' = x_1' \\ x_3 &= x'' = (x')' = x_2' \end{aligned} \quad \Rightarrow \quad \begin{aligned} x_1' &= x_2 \\ x_2' &= x_3 \end{aligned}$$

4. Find last eqn by subbing into original.

$$x_3' = x^{(3)} = \sin 2t - 3x'' - 2x' + 5x$$

$$x_3' = \sin 2t - 3x_3 - 2x_2 + 5x_1$$

Put it all Together

$$\begin{cases} x_1' = x_2 \\ x_2' = x_3 \\ x_3' = \sin 2t - 3x_3 - 2x_2 + 5x_1 \end{cases}$$

<u>EX</u> $x'' = x^3 + (x')^3$ second order non linear ... can't solve!

Transform into a system of ODE's...

$$\begin{aligned} x_1 &= x \\ x_2 &= x' = x_1' \end{aligned} \quad \Rightarrow \quad x_1' = x_2$$

$$x_2' = x'' = f(x, x') \quad \Rightarrow \quad x_2' = x_1^3 + (x_2)^3$$

First Order system

$$\begin{cases} x_1' = x_2 \\ x_2' = x_1^3 + x_2^3 \end{cases}$$

- We can solve this numerically... but we can also get LOTS of information from it!

INFORMATION WITHOUT SOLVING!!

Fixed Points – (x,y) points in the solution where nothing happens.
– where the rate of change = zero.

$$x_1' = x_2 \qquad \text{Fixed Point} \qquad x_1' = 0$$
$$x_2' = x_1^3 + x_2^3 \qquad\qquad x_2' = 0$$

Just Plug in and Solve: $0 = x_2$
$0 = x_1^3 + x_2^3$

$(0,0)$ is a fixed point.

- if our system started at exactly $(0,0)$ nothing would ever happen!

Direction Fields – given a 2-D system such that

$$\left.\begin{aligned} x_1' &= f(x_1, y_1) \\ y_1' &= g(x_1, y_1) \end{aligned}\right\} \text{independent variable does not appear!}$$

We can plot a direction field!

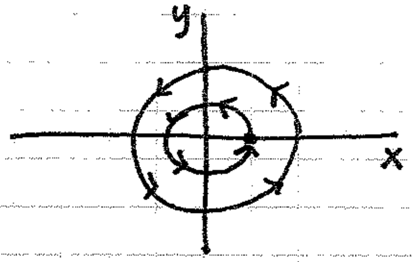
- At each point in space (x_1, y_1) plug into ODE to get (x_1', y_1')
  ↳ this gives a velocity field
  - direction of arrow gives you the path of your soln
  - length of the arrow gives you the velocity or speed of soln.

PPlane – online!

$$\left.\begin{aligned} x' &= y \\ y' &= x^3 + y^3 \end{aligned}\right\} \text{this is our same system}$$

$x_1 = x \quad x_2 = y.$

<u>Ex</u> $x' = -2y$
$y' = \frac{1}{2}x$



PLOT ON PPLANE.

Fixed Point $= (0,0)$

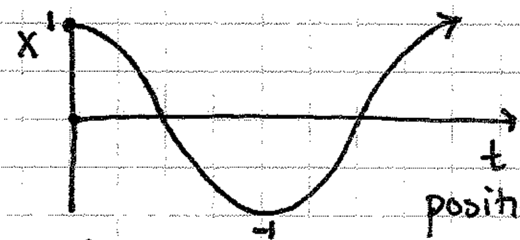
If I start at $(0,0)$ what does my soln do? Nothing!

If I start at $(1,0)$ — as time goes on my particle cycles about the origin.

How do we interpret this ... $x \Rightarrow$ position of my particle @ time t

$y \Rightarrow$ velocity of my particle @ time t

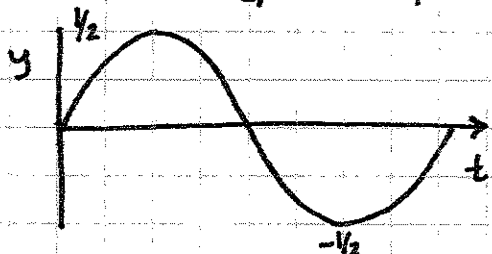
POSITION



position is oscillating

VELOCITY

$y = -\frac{1}{2}x'$



also oscillating...

(LETS FIND THE SOLN ...)

- put the eqn back into regular form...

If $x' = -2y$ then $x'' = -2y'$

and $y' = \frac{1}{2}x$ so $x'' = 2(\frac{1}{2}x) = x$

our original ode: $\boxed{x'' - x = 0}$ second order linear homogeneous...

$x(t) = A\sin t + B\cos t$ $\quad x(0) = 1$, $x'(0) = 0$

Then $\boxed{x(t) = \cos t}$ and $x'(t) = -\sin(t)$

This matches what we see in PPlane!

NOTE $y = -\frac{1}{2}x'$ so $\boxed{y(t) = \frac{1}{2}\sin(t)}$

Ex Transform the system into first order

$$\begin{cases} 2x'' = -6x + 2y \\ y'' = 2x - 2y + 40\sin 3t \end{cases}$$

use the same method as before ... do each one separately.

$x_1 = x$ | $y_1 = y$
$x_2 = x' = x_1'$ | $y_2 = y' = y_1'$
$x_2' = x'' = -3x + y$ | $y_2' = y'' = 2x - 2y + 40\sin 3t$

$x_1' = x_2$ | $y_1' = y_2$
$x_2' = -3x_1 + y_1$ | $y_2' = 2x_1 - 2y_1 + 40\sin 3t$

Four Equations and Four Unknowns!

Ex Solve the System $\begin{array}{l} x' = y \\ y' = 2x + y \end{array}$ for $x(t)$, $y(t)$

1. Find x'' $\quad x'' = y'$ sub in ...

$$x'' = 2x + y \quad \text{sub in again...}$$

$$x'' = 2x + x'$$

so $x'' - x' - 2x = 0$

solve like normal ...

$$r^2 - r - 2 = 0$$

$$(r+1)(r-2) = 0 \qquad r = -1, 2$$

$$\boxed{x(t) = Ae^{-t} + Be^{2t}}$$

2. Then go back an solve for y ...

$$y = x' \quad \text{so} \quad \boxed{y(t) = -Ae^{-t} + 2Be^{2t}}$$

IF TIME

Transform into a first order system ... $x'' + 3x' + 7x = t^2$

SOLVE FOR $(x(t), y(t))$ $\begin{cases} x' = y \\ y' = -x \end{cases}$