- ~~Hand in HW 20 — Test Corrections.~~
- ~~HW 21 due Mon~~
- ~~HW 22 due Wed.~~

- We are going to move to more advanced methods for solving systems of eqns ... to do this we need matrices.

MATRIX REVIEW — more info online if you need more review → be prepared for matrix operation on monday!

An $m \times n$ matrix → m rows, n columns

$$A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ a_{21} & & \vdots \\ \vdots & & \vdots \\ a_{m1} & & a_{mn} \end{bmatrix}$$

zero matrix — all entries are zero

identity matrix — diagonal entries are ones everything else is zero.

Addition or Subtractions → add or subtract the corresponding ellements.

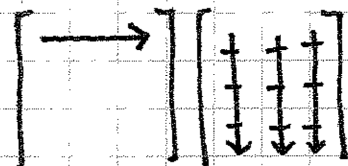
1. Commutatire $A + B = B + A$
2. Associatire $A + (B+C) = (A+B) + C$
3. Distributative $(c+d)A = cA + dA$ where c and d are scalars.

The Transpose → switch the rows and columns.

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \quad \text{then} \quad A^T = \begin{bmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{bmatrix}$$

Column Vector → $b = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$ Row Vector → $a = [a_1, a_2, a_3]$

Matrix Multiplication: AB — only defined if A has the same # of columns as B has rows!

In my mind: 

repeat for each row of A.

$$C = AB \quad \text{then} \quad c_{ij} = \sum_{k=1}^{p} a_{ik} b_{kj}$$

$$A = \begin{bmatrix} 2 & -3 \\ -1 & 5 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 2 \\ 4 & 1 \end{bmatrix}$$

then $$AB = \begin{bmatrix} 2 & -3 \\ -1 & 5 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 4 & 1 \end{bmatrix} = \begin{bmatrix} 2\cdot1 + -3\cdot4 & 2\cdot2 + -3\cdot1 \\ -1\cdot1 + 5\cdot4 & -1\cdot2 + 5\cdot1 \end{bmatrix}$$

$$= \begin{bmatrix} 2-12 & 4-3 \\ -1+20 & -2+5 \end{bmatrix} = \begin{bmatrix} -10 & 1 \\ 19 & 3 \end{bmatrix}$$

Properties:
1. Associative $A(BC) = (AB)C$
2. Distributative $A(B+C) = AB + AC$

BUT $AB \neq BC$ and $AB = \emptyset$ even if $A \neq 0$ and $B \neq 0$

We already know the DETERMINANT ... make sure you can calculate this.

Now we need to apply all these rules to functions!

Matrix Valued Functions

$$\vec{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix} \quad \text{or} \quad A(t) = \begin{bmatrix} a_{11}(t) & \cdots & a_{1n}(t) \\ \vdots & & \vdots \\ a_{m1}(t) & \cdots & a_{mn}(t) \end{bmatrix}$$

<u>aside</u> $\vec{x}$ is a vector
if A is square $m=n$ it is a tensor.
Tensors arise by taking the gradient of a vector...

We can take derivatives of these!

$$\vec{x}'(t) = \frac{d\vec{x}}{dt} = \begin{bmatrix} dx_1(t)/dt \\ dx_2(t)/dt \\ \vdots \end{bmatrix}$$

take the derivative of the components separately.

<u>Ex</u> $\vec{x}(t) = \begin{bmatrix} t \\ t^2 \\ e^{-t} \end{bmatrix}$ then $\vec{x}'(t) = \begin{bmatrix} 1 \\ 2t \\ -e^{-t} \end{bmatrix}$

we have all our rules of derivatives...

$$\frac{d}{dt}(A+B) = \frac{dA}{dt} + \frac{dB}{dt}$$

$$\frac{d}{dt}AB = A\frac{dB}{dt} + B\frac{dA}{dt}$$

<u>Ex</u> Write the system in matrix form

$$\begin{aligned} x_1' &= 4x_1 - 3x_2 \\ x_2' &= 6x_1 - 7x_2 \end{aligned} \rightarrow \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}' = \frac{d\vec{x}}{dt} = \begin{bmatrix} 4 & -3 \\ 6 & -7 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

or $$\frac{d\vec{x}}{dt} = \begin{bmatrix} 4 & -3 \\ 6 & -7 \end{bmatrix}\vec{x}$$

now verify that $\vec{x}_1(t) = \begin{bmatrix} 3e^{2t} \\ 2e^{2t} \end{bmatrix}$ and $\vec{x}_2(t) = \begin{bmatrix} e^{-5t} \\ 3e^{-5t} \end{bmatrix}$

are solutions

Just take the derivatives and sub in ...

$$\vec{X}_1' = \begin{bmatrix} 6e^{2t} \\ 4e^{2t} \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 4 & -3 \\ 6 & -7 \end{bmatrix} \vec{X}_1 = \begin{bmatrix} 4 & -3 \\ 6 & -7 \end{bmatrix} \begin{bmatrix} 3e^{2t} \\ 2e^{2t} \end{bmatrix}$$

$$= \begin{bmatrix} 12e^{2t} - 6e^{2t} \\ 18e^{2t} - 14e^{2t} \end{bmatrix} = \begin{bmatrix} 6e^{2t} \\ 4e^{2t} \end{bmatrix} \checkmark$$

so $\vec{X}_1' = \begin{bmatrix} 4 & -3 \\ 6 & -7 \end{bmatrix} \vec{X}_1$ $\quad \vec{X}_1$ is a soln.

YOU SHOW THAT $\vec{X}_2$ IS a soln !!

GENERAL FORM FOR FIRST ORDER LINEAR SYSTEM:

$$\frac{d\vec{x}}{dt} = P(t)\vec{x} + \vec{f}(t)$$

non homogeneous term

part of the homogeneous soln.

So we still have all the same rules and theorms as ~~[illegible]~~ higher order linear ODE's:

1. We begin by considering the homogeneous soln:

$$\frac{d\vec{x}}{dt} = P(t)\vec{x}$$

if $\vec{x}$ has n components then we expect n-solutions

2. Principle of superposition if $\vec{X}_1, \vec{X}_2, \vec{X}_3 \ldots$ are all solutions

Then the general soln is the sum of these

$$\vec{X} = c_1\vec{X}_1 + c_2\vec{X}_2 + c_3\vec{X}_3 \ldots$$

NOTE: $\vec{X}_1$ $\vec{X}_2$ etc must be linearly independent !

3. We can show linear Independence of solutions!

$$W(t) = \text{Wronskian} = \begin{bmatrix} | & | & & | \\ \vec{x}_1(t) & \vec{x}_2(t) & \cdots & \vec{x}_n(t) \\ | & | & & | \end{bmatrix}$$

write solns along column

if $\det(W) = 0$ then $\vec{x}_n$'s are linearly dependent
if $\det(W) \neq 0$ then $\vec{x}_n$'s are linearly independent

<u>EX</u> $\vec{x}(t) = \cancel{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ satisfies $\vec{x}\,'(t) = A\vec{x}(t)$

where A is 3×3 matrix.

How many LI solns do we expect? Why?

Assume we found: $\vec{x}_1 = \begin{bmatrix} 2e^{t} \\ 2e^{t} \\ e^{t} \end{bmatrix}$ $\vec{x}_2 = \begin{bmatrix} 2e^{3t} \\ 0 \\ -e^{3t} \end{bmatrix}$ $\vec{x}_3 = \begin{bmatrix} 2e^{5t} \\ -2e^{5t} \\ e^{5t} \end{bmatrix}$

Show that these are Linearly independent and write down the general soln.

$$\det(W) = -16e^{9t} \neq 0$$

$$\vec{x}(t) = c_1\vec{x}_1 + c_2\vec{x}_2 + c_3\vec{x}_3$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = c_1\begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}e^{t} + c_2\begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix}e^{3t} + c_3\begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}e^{5t}$$

<u>or</u>

$$x = 2c_1e^{t} + 2c_2e^{3t} + 2c_3e^{5t}$$

$$y = 2c_1e^{t} - 2c_3e^{5t}$$

$$z = c_1e^{t} - c_2e^{3t} + c_3e^{5t}$$

NOTE ... our goal next week find $e^{\lambda t}$ and $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$ to construct these solns!!