[redacted]

Lecture 26 Differential Eqns.

The Eigenvalue Method for Homogeneous Systems.

LAST TIME ... review of matrices and we saw solutions of the form:

$$\vec{x} = c_1 \underbrace{\begin{bmatrix} \} \\ \} \end{bmatrix}}_{\vec{v}_1} e^{\lambda_1 t} + c_2 \underbrace{\begin{bmatrix} \} \\ \} \end{bmatrix}}_{\vec{v}_2} e^{\lambda_2 t} + \cdots$$

So lets move forward by assuming that for constant coefficient eqns $\vec{x} = \vec{v}e^{\lambda t}$

a guess lets see what happens...

$$\frac{d\vec{x}}{dt} = A\vec{x}$$ A contains all of our constant coefficients.

assume $\vec{x} = \vec{v}e^{\lambda t}$ then $\frac{d\vec{x}}{dt} = \vec{v}\lambda e^{\lambda t}$

so $\vec{v}\lambda e^{\lambda t} = A\vec{v}e^{\lambda t} \rightarrow \vec{v}\lambda = A\vec{v}$

so if we can find a λ and $\vec{v}$ that satisfy this we have a solution!

rewrite this eqn: $\vec{v}\lambda - A\vec{v} = 0$

$A\vec{v} - \lambda\vec{v} = 0$

$(A - \lambda I)\vec{v} = 0$ for this to be true $|(A - \lambda I)| = 0$

<u>DEFINE</u> The number $\underline{\lambda}$ either zero or nonzero is called an eigenvalue of the matrix A found from $|A - \lambda I| = 0$

The eigenvector $\underline{\vec{v}}$ associated with λ is a non-zero vector such that

$$(A - \lambda I)\vec{v} = 0$$

Eigenvalue Solns of $\vec{x}' = A\vec{x}$

given an eigenvector and eigenvalue the soln takes the form

$$\vec{x} = \vec{v}e^{\lambda t}$$

THE METHOD

1. Solve $|A - \lambda I| = 0$ for all possible eigenvalues
   call this the characteristic eqn.

   $$\lambda_1, \lambda_2, \ldots, \lambda_n$$

2. Find n linearly independent eigenvectors associated with each eigenvalue

   $$\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$$

   where we find these by plugging <u>each</u> eigenvalue into the eqn
   $$(A - \lambda I)\vec{v} = 0$$
   and finding $\vec{v}$.

3. Step two is not always possible, but when it is we get n linearly independent solutions and write the general soln:

   $$\vec{x} = c_1\vec{v}_1 e^{\lambda_1 t} + c_2\vec{v}_2 e^{\lambda_2 t} + \cdots + c_n\vec{v}_n e^{\lambda_n t}$$

4. Then apply initial conditions if given.

CASES:
1. Distinct Real Eigenvalues
2. Complex Eigenvalues
3. Repeated Eigenvalues.

If the eigenvalues are distinct then we are garanteed n linearly independent eigenvectors.
and our soln in $X(t) = c_1\vec{v}_1 e^{\lambda_1 t} + c_2\vec{v}_2 e^{\lambda_2 t} + \cdots$

Ex $\quad x_1' = 4x_1 + 2x_2$
$\quad x_2' = 3x_1 - x_2$

write in matrix form: $\vec{x}\,' = \begin{bmatrix} 4 & 2 \\ 3 & -1 \end{bmatrix} \vec{x}$ where $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

1. Find eigenvalues:

$$|A - \lambda I| = \left| \begin{bmatrix} 4 & 2 \\ 3 & -1 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \right| = \begin{vmatrix} 4-\lambda & 2 \\ 3 & -1-\lambda \end{vmatrix}$$

usually skip to here!

$$= (4-\lambda)(-1-\lambda) - 6 = -4 - 3\lambda + \lambda^2 - 6 = \lambda^2 - 3\lambda - 10 = 0$$

$$(\lambda - 5)(\lambda + 2) = 0 \quad \lambda_1 = 5 \quad \lambda_2 = -2$$

real distinct e-vals.

2. Plug each eigenvalue into $(A - \lambda I)\vec{v} = 0$

$\lambda_1 = 5$ then $A - \lambda I = \begin{bmatrix} 4-5 & 2 \\ 3 & -1-5 \end{bmatrix} = \begin{bmatrix} -1 & 2 \\ 3 & -6 \end{bmatrix}$

$$(A - \lambda I)\vec{v} = 0 \Rightarrow \begin{bmatrix} -1 & 2 \\ 3 & -6 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

or $\left.\begin{matrix} -a + 2b = 0 \\ 3a - 6b = 0 \end{matrix}\right\}$ notice that these are constant multiples

both: $a = 2b$ eigenvectors are not unique.

choose $b = 1$ then $a = 2$

$$\vec{v}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$\lambda_2 = -2$ then $A - \lambda I = \begin{bmatrix} 4+2 & 2 \\ 3 & -1+2 \end{bmatrix} = \begin{bmatrix} 6 & 2 \\ 3 & 1 \end{bmatrix}$

$$\begin{bmatrix} 6 & 2 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad \begin{matrix} 6a + 2b = 0 \\ 3a + b = 0 \end{matrix} \Rightarrow \begin{matrix} 3a = -b \\ \text{if } a = 1 \quad b = -3 \end{matrix}$$

$$\vec{v}_2 = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$$

3. Build the solution using eiginvalues and vectors

$$\vec{x} = c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{5t} + c_2 \begin{bmatrix} 1 \\ -3 \end{bmatrix}$$

You Try: $\left.\begin{aligned} x_1' &= x_1 + 2x_2 \\ x_2' &= 2x_1 + x_2 \end{aligned}\right\rangle \quad x(t) = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{3t} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-t}$

Even if eiginvalues are complex as long as they are distinct we still get n linearly dependent solns.

Assume we get $\lambda_1 = p + qi$ and $\lambda_2 = p - qi$

then $\lambda_2 = \overline{\lambda_1}$ complex conjugate pairs

we would find that the associated eigenvectors would also be complex conjugate pairs.

$\vec{V}_1 = \vec{a} + \vec{b}i$ and $\vec{V}_2 = \vec{a} - \vec{b}i$

$$\vec{V}_2 = \overline{\vec{V}_1}$$

- This means that once we have one eigenvector we have the other — so we only solve for one vector per complex conjugate pair.
- We can use $e^{iqt} = \cos qt + i \sin qt$ to write our solution as sines and cosines
- We express the parts of the soln as the real and imaginary parts of the one vector we find.

Ex $\left.\begin{aligned} x_1' &= 4x_1 - 3x_2 \\ x_2' &= 3x_1 + 4x_2 \end{aligned}\right\}$ $\vec{x}' = \begin{bmatrix} 4 & -3 \\ 3 & 4 \end{bmatrix}\vec{x}$

1. Find the eigenvalues ...

$$\begin{vmatrix} 4-\lambda & -3 \\ 3 & 4-\lambda \end{vmatrix} = (4-\lambda)^2 + 9 = 0$$

$$= 16 - 8\lambda + \lambda^2 + 9 = \lambda^2 - 8\lambda + 25 = 0$$

$$\therefore \lambda = \frac{8 \pm \sqrt{64-100}}{2} = \frac{4 \pm \sqrt{-36}}{2}$$
$$= 4 \pm 3i$$

2. Find one eigenvector $\overset{\text{use}}{\wedge}$ $\lambda = 4 + 3i$

$$\begin{bmatrix} 4-4-3i & -3 \\ 3 & 4-4-3i \end{bmatrix} = \begin{bmatrix} 3i & -3 \\ 3 & -3i \end{bmatrix} \Rightarrow \begin{bmatrix} -3i & -3 \\ 3 & -3i \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = 0$$

so $-3ia - 3b = 0$ $\quad a = -ib$

$3a - 3ib = 0$

let $a = 1$

then $b = i$

$$v_1 = \begin{bmatrix} 1 \\ i \end{bmatrix}$$

Technically our soln: $x(t) = c_1\begin{bmatrix} 1 \\ i \end{bmatrix}e^{(4+3i)t} + c_2\begin{bmatrix} 1 \\ -i \end{bmatrix}e^{(4-3i)t}$

but we can write this in a nicer form: Sine and Cosine!

3. Consider $v_1 e^{\lambda_1 t} = \begin{bmatrix} 1 \\ i \end{bmatrix}e^{4t}e^{+3it} = \begin{bmatrix} 1 \\ i \end{bmatrix}e^{4t}(\cos 3t + i\sin 3t)$

$$= e^{4t}\begin{bmatrix} 1 \\ i \end{bmatrix}(\cos 3t + i\sin 3t) = e^{4t}\begin{bmatrix} \cos 3t + i\sin 3t \\ i\cos 3t - \sin 3t \end{bmatrix}$$

now we separate into real and imaginary parts.

$$= e^{4t}\begin{bmatrix}\cos 3t\\ -\sin 3t\end{bmatrix} + ie^{4t}\begin{bmatrix}\sin 3t\\ \cos 3t\end{bmatrix}$$

use the real part of this as $\vec{x}_1(t)$
and the imaginary part as $\vec{x}_2(t)$

$$\boxed{x(t) = c_1 e^{4t}\begin{bmatrix}\cos 3t\\ \sin 3t\end{bmatrix} + c_2 e^{4t}\begin{bmatrix}\sin 3t\\ -\cos 3t\end{bmatrix}}$$

NOTE We would end up with the same thing if we wrote

$$x(t) = c_1\vec{v}_1 e^{\lambda_1 t} + c_2\vec{v}_2 e^{\lambda_2 t}$$

then simplified like crazy!

much easier to find just $\vec{v}_1$ and look for the real and imaginary parts!

YOU TRY: $\left.\begin{aligned} x_1' &= x_1 - 2x_2\\ x_2' &= 2x_1 + x_2\end{aligned}\right\}$ $\quad \vec{x}(t) = e^{t}\left[c_1\begin{bmatrix}\cos 2t\\ \sin 2t\end{bmatrix} + c_2\begin{bmatrix}-\sin 2t\\ \cos 2t\end{bmatrix}\right]$