- ~~Hand in 22~~
- ~~23 Due Friday~~
- ~~24 Due Monday.~~

Last Time: Eigenvalue Method - Homogeneous Systems.

$$\vec{x}' = A\vec{x}$$

1. Find eigenvalues of A: $|A-\lambda I|=0$ then $\lambda_1, \lambda_2 \ldots$ e-vals
2. Find associated eigenvectors: $(A-\lambda I)\vec{v}=0$ then $v_1, v_2 \ldots$
3. Write down the soln

$$x(t) = c_1\vec{v}_1 e^{\lambda_1 t} + c_2\vec{v}_2 e^{\lambda_2 t} + \ldots$$

NOTE: with complex roost find one evec the separate $\vec{v}_1 e^{\lambda_1 t}$ into real and imaginary parts.

$$x(t) = c_1 \operatorname{Re}\{\vec{v}_1 e^{\lambda_1 t}\} + c_2 \operatorname{Im}\{\vec{v}_1 e^{\lambda_1 t}\}$$

Today we deal with the case of repeated roots.

If $|A-\lambda I|=0$ has at least one repeated root the we have two cases:

COMPLETE REPEATED EIGENVALUES - good news case, even though λ is repeated we can still find multiple linearly independent eigenvectors an construct our system

DEFECTIVE REPEATED EIGENVALUE - if we cannot find enough linearly independent eigenvectors then we must construct them. These are called generalized eigenvectors.

Ex

$$\vec{x}' = \begin{bmatrix} 9 & 4 & 0 \\ -6 & -1 & 0 \\ 6 & 4 & 3 \end{bmatrix} \vec{x}$$

1. Characteristic Eqn $|A - \lambda I| = 0$

use cofactors to do determinant of a 3x3.

$$\begin{vmatrix} 9-\lambda & 4 & 0 \\ -6 & -1-\lambda & 0 \\ 6 & 4 & 3-\lambda \end{vmatrix} = (9-\lambda)\begin{vmatrix} -1-\lambda & 0 \\ 4 & 3-\lambda \end{vmatrix} - 4\begin{vmatrix} -6 & 0 \\ 6 & 3-\lambda \end{vmatrix} + 0$$

$$= (9-\lambda)(-1-\lambda)(3-\lambda) - 4(-6)(3-\lambda)$$

look for things to factor first...

$$= (3-\lambda)\left[(9-\lambda)(-1-\lambda) + 24\right] = (3-\lambda)\left[-9 - 8\lambda + \lambda^2 + 24\right]$$

$$= (3-\lambda)\left[\lambda^2 - 8\lambda + 15\right] = (3-\lambda)(\lambda-5)(\lambda-3)$$

so $\lambda = 3, 3, 5$ repeated e-val.

2. Consider e-vals separately to find e-vecs.

$\lambda = 5$

$$\begin{bmatrix} 4 & 4 & 0 \\ -6 & -6 & 0 \\ 6 & 4 & -2 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left.\begin{aligned} 4a + 4b &= 0 \\ -6a - 6b &= 0 \end{aligned}\right\} \quad a = -b$$

let $a = 1$ then $b = -1$

$$6a + 4b - 2c = 0 \;\} \quad c = +3a + 2b \qquad c = 1$$

$$\vec{v}_1 = \begin{bmatrix} 1 & -1 & 1 \end{bmatrix}^T$$

$\lambda = 3$

$$\begin{bmatrix} 6 & 4 & 0 \\ -6 & -4 & 0 \\ 6 & 4 & 0 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{aligned} 6a + 4b &= 0 \\ a &= -\tfrac{2}{3}b \end{aligned}$$

but c is never specified so we take two cases for c: $c = 1$ and $c = 0$

Assume $c=0$ and choose values for a and b:

let $b=3$ then $a=-2$ $\quad \vec{v}_2 = [-2 \;\; 3 \;\; 0]^T$

Assume $c=1$ and $a=b=0$ This still satisfies the eqns!

$$\vec{v}_3 = [0 \;\; 0 \;\; 1]^T$$

$$\vec{x}(t) = c_1 e^{5t}\begin{bmatrix}1\\-1\\1\end{bmatrix} + c_2 e^{3t}\begin{bmatrix}0\\0\\1\end{bmatrix} + c_3 e^{3t}\begin{bmatrix}-2\\+3\\0\end{bmatrix}$$

This was the good news case... even though we had repeated e-vals we were able to construct enough e-vectors.

<u>DEFECTIVE CASE</u>

consider $A = \begin{bmatrix}1 & -3\\3 & 7\end{bmatrix}$ then $|A-\lambda I| = (1-\lambda)(7-\lambda)+9$

$$= 7-8\lambda+\lambda^2+9$$
$$= \lambda^2 - 8\lambda + 16 = (\lambda-4)^2$$

$\lambda = 4,4$ repeated $\quad (A-\lambda I)\vec{v}_1 = 0$

$$\begin{bmatrix}-3 & -3\\3 & 3\end{bmatrix}\begin{bmatrix}a\\b\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix} \qquad 3a+3b=0 \text{ or } a=-b$$

once we specify a we know b: $\vec{v}_1 = [1 \;\; -1]^T$ and $v \neq [0,0]$

$\lambda = 3$ is a defective eigenvalue and we need a way to find eigenvectors.

Recall: When we had repeated roots before... constructed solns like $(c_1 + c_2 t)e^{rt}$

lets try this here...

We know $\vec{x} = \vec{v}_1 e^{\lambda t}$ is a soln ...

lets try $\vec{x} = (\vec{v}_1 t + \vec{v}_2) e^{\lambda t}$ as our soln and see what happens.

$\vec{x}' = \vec{v}_1 e^{\lambda t} + (\vec{v}_1 t + \vec{v}_2)\lambda e^{\lambda t}$ so $\vec{x}' = A\vec{x}$

$\vec{v}_1 e^{\lambda t} + (v_1 t + \vec{v}_2)\lambda e^{\lambda t} = A(\vec{v}_1 t + \vec{v}_2) e^{\lambda t}$ distribute...

$(\vec{v}_1 + \vec{v}_2 \lambda) e^{\lambda t} + \vec{v}_1 \lambda t e^{\lambda t} = A v_2 e^{\lambda t} + A\vec{v}_1 t e^{\lambda t}$ like terms must be equal

$\vec{v}_1 + \vec{v}_2 \lambda = A\vec{v}_2$ and $\vec{v}_1 \lambda = A\vec{v}_1$

$\vec{v}_1 = (A - \lambda I)\vec{v}_2$

this gives a relationship between $\vec{v}_1$ and $\vec{v}_2$.

$(A - \lambda I)\vec{v}_1 = 0$ we already solve this!

Need to find a way to use these... multiply through by $(A - \lambda I)$ ...

$(A - \lambda I)\vec{v}_1 = (A - \lambda I)^2 \vec{v}_2 \Rightarrow 0 = (A - \lambda I)^2 \vec{v}_2$

so we can use this to solve for $\vec{v}_2$ and then

$\vec{v}_1 = (A - \lambda I)\vec{v}_2$

From before $\lambda = 4, 4$
we found only $\vec{v}_1 = [1 \ -1]^T$
this was not enough!

Instead find $(A - \lambda I)^2 = \begin{bmatrix} -3 & -3 \\ 3 & 3 \end{bmatrix}\begin{bmatrix} -3 & -3 \\ 3 & 3 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ ANY CHOICE of $\vec{v}_2$ will work it just has to be nonzero

This will always happen for simple cases
BUT CHECK!

Choose: $\vec{v}_2 = [1 \ 0]^T$

I always choose this...

The multiply by $(A-\lambda I)$ to get $\vec{v}_1$ that matches this choice...

$$\vec{v}_1 = \begin{bmatrix} -3 & -3 \\ 3 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -3 \\ 3 \end{bmatrix} \quad \text{so} \quad \vec{v}_1 = [-3 \ \ 3]^T$$

it is a constant multiple of our original guess... but that constant is forced by our $\vec{v}_2$ choice

OUR SOLN:

$$x(t) = c_1 \vec{v}_1 e^{\lambda t} + c_2 (\vec{v}_1 t + \vec{v}_2) e^{\lambda t}$$

$$x(t) = c_1 \begin{bmatrix} -3 \\ 3 \end{bmatrix} e^{4t} + c_2 \begin{bmatrix} -3t+1 \\ 3t \end{bmatrix} e^{4t}$$

<u>The general method</u> $\quad \vec{x}' = A\vec{x}$

Check $(A-\lambda I)u_1 = 0$ to see how many vectors you are missing.

Now successively multiply:

Form a "chain" of eigenvectors.

For higher order $(v_3 + v_2 t + v_1 t^2 + \dots)$

$$(A-\lambda I)u_1 = u_2 \neq 0$$
$$(A-\lambda I)u_2 = u_3 \neq 0$$
$$\vdots$$
$$(A-\lambda I)u_{n-1} = u_n \neq 0 \quad \underline{\text{and}} \quad (A-\lambda I)u_n = 0$$

This lets us choose $\vec{u}_n$ to be what ever we want and our $u_1 \ u_2 \ u_3 \dots$ form a chain of generalized eigenvectors that may form our soln.

IN PRACTICE: $\quad (A-\lambda I)^n \vec{v}_n = 0 \quad$ where n is the total # of vectors you hope to generate.

usually expect $\quad (A-\lambda I)^n = \vec{\emptyset} \quad$ [redacted]. ? Check this.

Choose $\vec{v}_n$ to be whatever then solve

or pick v based on $(A-\lambda I)^n v_n = 0$

$$\vec{v}_{n-1} = (A-\lambda I)\vec{v}_n$$
$$\vec{v}_{n-2} = (A-\lambda I)\vec{v}_{n-1}$$
$$\vdots$$
$$\vec{v}_1 = (A-\lambda I)\vec{v}_2$$

our linearly independent solns are written:

$$\vec{x}_1 = v_1 e^{\lambda t} \qquad \vec{x}_2 = (v_1 t + v_2) e^{\lambda t} \quad \dots$$

$$\vec{x}_n = (v_1 t^{n-1} + v_2 t^{n-2} + \dots + v_n) e^{\lambda t}$$

this is the one you just chose!

Ex $\vec{x}' = \begin{bmatrix} 0 & 1 & 2 \\ -5 & -3 & -7 \\ 1 & 0 & 0 \end{bmatrix} \vec{x}$

$$|A - \lambda I| = \begin{vmatrix} -\lambda & 1 & 2 \\ -5 & -3-\lambda & -7 \\ 1 & 0 & -\lambda \end{vmatrix} = (\lambda+1)^3 = 0 \qquad \lambda = -1 \text{ defective with mult 3.}$$

after simplifying!!

Try to find an eigenvector:

$$\begin{bmatrix} 1 & 1 & 2 \\ -5 & -2 & -7 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad \begin{aligned} a + b + 2c &= 0 \\ -5a - 2b - 7c &= 0 \\ a + c &= 0 \end{aligned}$$

so $c = -a$ then
$b - a = 0$ and $b = a$

if we pick one value we are forced to pick the others!

$v_1 = [a \;\; a \;\; -a]^T \quad a \neq 0.$

need more vectors than this...

We know that

$(A - \lambda I)^3 = \vec{\emptyset}$ → Try this it will work!!

because we need 3 vectors.

- only fails if you get two out of three repeated vectors

$(A - \lambda I)^3 \vec{v}_3 = 0 \qquad$ let $\vec{v}_3 = [1, 0, 0]^T$

this can be anything!

Then $$\vec{V}_2 = \underbrace{\begin{bmatrix} 1 & 1 & 2 \\ -5 & -2 & -7 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}}_{(A-\lambda I)\vec{V}_3} = \begin{bmatrix} 1 \\ -5 \\ 1 \end{bmatrix}$$

And $$\vec{V}_1 = \begin{bmatrix} 1 & 1 & 2 \\ -5 & -2 & -7 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -5 \\ 1 \end{bmatrix} = \begin{bmatrix} 1-5+2 \\ -5+10-7 \\ 1+0+1 \end{bmatrix} = \begin{bmatrix} -2 \\ -2 \\ 2 \end{bmatrix}$$

this matches $V_1 = [a \;\; a \;\; -a]^T$ when $a = -2$!!

our soln

$$\vec{X}_1 = \begin{bmatrix} -2 \\ -2 \\ -2 \end{bmatrix} e^{-t}$$

$$\vec{X}_2 = \begin{bmatrix} -2t+1 \\ -2t-5 \\ -2t+1 \end{bmatrix} e^{-t}$$

$$\vec{X}_3 = \begin{bmatrix} -\cancel{2}t^2 + t + 1 \\ -\cancel{2}t^2 - 5t \\ -\cancel{2}t^2 + t \end{bmatrix} e^{-t}$$

$$\vec{X}(t) = c_1\vec{X}_1 + c_2\vec{X}_2 + c_3\vec{X}_3$$

$$X_1(t) = V_1 e^{\lambda t}$$

$$X_2(t) = (V_1 t + V_2) e^{\lambda t}$$

$$X_3(t) = \left(\tfrac{1}{2}V_1 t^2 + V_2 t + V_3\right) e^{\lambda t}$$

$$X_k(t) = \left(\frac{V_1 t^{k-1}}{(k-1)!} + \cdots + V_{k-1} t + V_k\right) e^{\lambda t}.$$