The method of undetermined Coeficients for Systems:

5.6

1. Solve for the Homogeneous Part $\vec{x}_H' = A\vec{x}_H$ using eigenvalue technique.
2. Make a guess for $\vec{x}_p$ based on $\vec{f}(t)$
3. Plug your guess into the eqn $\vec{x}_p' = A\vec{x}_p + \vec{f}(t)$ an solve for the unknown coeficients.
4. Final answer:

$$\vec{x}(t) = \vec{x}_H(t) + \vec{x}_p(t)$$

~~Lets look for~~ ~~V // constant and solve this system.~~

EXAM 3 Wed 4/5

IN TWO WEEK

Ex $x' = x + 2y + 3$
$y' = 2x + y - 2$

$$\vec{x}\,' = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \vec{x} + \begin{bmatrix} 3 \\ -2 \end{bmatrix}$$

1. Homogeneous Soln: $\vec{x}_H = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{3t} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-t}$

   – do this just like normal!

2. Non homogeneous.

   Consider $f(t) = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$ make a guess for the solution that looks like this.

   $\vec{x}_p = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$ plug this into the matrix ODE...

$$x_p' = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad A\vec{x}_p = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} a_1 + 2a_2 \\ 2a_1 + a_2 \end{bmatrix}$$

This is exactly like MUD from before we are just using matrices!

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} a_1 + 2a_2 \\ 2a_1 + a_2 \end{bmatrix} + \begin{bmatrix} 3 \\ -2 \end{bmatrix}$$

$a_1 + 2a_2 = -3$
$2a_1 + a_2 = 2$

solve for a_1 and a_2 $\qquad a_1 = -3 - 2a_2$

$$2[-3 - 2a_2] + a_2 = 2$$
$$-6 - 3a_2 = 2$$
$$-3a_2 = 8 \qquad a_2 = -8/3$$

then $a_1 = -3 + \frac{16}{3} = \frac{7}{3}$

$$\vec{x}_p = \frac{1}{3}\begin{bmatrix} 7 \\ -8 \end{bmatrix}$$

soln: $\vec{x} = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{3t} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-t} + \frac{1}{3}\begin{bmatrix} 7 \\ -8 \end{bmatrix}$

Lets try an intense example...

Ex
$$x' = x - 5y + 2\sin(t)$$
$$y' = x - y - 3\cos(t)$$

In matrix form:

$$\vec{x} = \begin{bmatrix} 1 & -5 \\ 1 & -1 \end{bmatrix} \vec{x} + \begin{bmatrix} 2\sin(t) \\ -3\cos(t) \end{bmatrix}$$

#1 Homogeneous soln $\vec{x}' = A\vec{x}$ drop the $f(t)$.

E-vals $|A - \lambda I| = \begin{vmatrix} 1-\lambda & -5 \\ 1 & -1-\lambda \end{vmatrix} = (1-\lambda)(-1-\lambda) + 5 = -1 + \lambda^2 + 5 = \lambda^2 + 4 = 0$

$$\lambda = \pm 2i$$

★ complex ... so choose one $\lambda = 2i$

Find associated vector $(A - \lambda I)v = 0$

$$\begin{bmatrix} 1-2i & -5 \\ 1 & -1-2i \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \emptyset \qquad \begin{aligned} (1-2i)a - 5b &= 0 \\ a - (1+2i)b &= 0 \end{aligned}$$

$$a = (1+2i)b \qquad v = \begin{bmatrix} 1+2i \\ 1 \end{bmatrix}$$

Check this also satisfies the other eqn...

$(1-2i)(1+2i) - 5$

$1 + 4 - 5 = 0$ ✓

Write soln as $c_1 \mathrm{Re} + c_2 \mathrm{Im}$:

$$\begin{bmatrix} 1+2i \\ 1 \end{bmatrix} e^{2it} = \begin{bmatrix} 1+2i \\ 1 \end{bmatrix} (\cos(2t) + i\sin(2t))$$

$$= \begin{bmatrix} \cos(2t) + i\sin(2t) + 2i\cos(2t) - 2\sin(2t) \\ \cos(2t) + i\sin(2t) \end{bmatrix}$$

$$= \begin{bmatrix} \cos(2t) - 2\sin(2t) \\ \cos(2t) \end{bmatrix} + i \begin{bmatrix} \sin(2t) + 2\cos(2t) \\ \sin(2t) \end{bmatrix}$$

Homogeneous soln:

$$\vec{X}_H(t) = c_1 \begin{bmatrix} \cos(2t) - \sin(2t) \\ \cos(2t) \end{bmatrix} + c_2 \begin{bmatrix} 2\cos(2t) + \sin(2t) \\ \sin(2t) \end{bmatrix}$$

#2 Non homogeneous soln — MUC

consider $f(t) = \begin{bmatrix} 2\sin(t) \\ -3\cos(t) \end{bmatrix}$

make a guess $\vec{X}_{NH} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} \sin(t) + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \cos(t)$

check for duplications! No Dup. YAY!

plug in and balance eqns...

$$\vec{X}'_{NH} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} \cos(t) - \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \sin(t)$$

$$A\vec{X}_{NH} = \begin{bmatrix} 1 & -5 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} \sin(t) + \begin{bmatrix} 1 & -5 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \cos(t)$$

$$= \begin{bmatrix} a_1 - 5a_2 \\ a_1 - a_2 \end{bmatrix} \sin(t) + \begin{bmatrix} b_1 - 5b_2 \\ b_1 - b_2 \end{bmatrix} \cos(t)$$

plug these in!

$$X'_{NH} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} \cos(t) - \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \sin(t) = A\vec{x}_{nh} + f(t) = \begin{bmatrix} a_1 - 5a_2 + 2 \\ a_1 - a_2 \end{bmatrix} \sin(t) + \begin{bmatrix} b_1 - 5b_2 \\ b_1 - b_2 - 3 \end{bmatrix} \text{co}$$

Match up the sides

$$-\begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} a_1 - 5a_2 + 2 \\ a_1 - a_2 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} b_1 - 5b_2 \\ b_1 - b_2 - 3 \end{bmatrix}$$

FOUR EQNS + FOUR UNKNOWNS

$$-b_1 - a_1 + 5a_2 = 2$$
$$-b_2 - a_1 + a_2 = 0$$
$$-b_1 + 5b_2 + a_1 = 0$$
$$-b_1 + b_2 + a_2 = -3$$

$$\begin{bmatrix} -1 & 0 & -1 & 5 \\ 0 & -1 & -1 & 1 \\ -1 & 5 & 1 & 0 \\ -1 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 0 \\ -3 \end{bmatrix}$$

ON SAGE MATH ... $A\vec{b} = \vec{c}$

$$\vec{b} = A^{-1}\vec{c}$$

$$\vec{b} = \begin{bmatrix} 17/3 \\ 1 \\ 2/3 \\ 5/3 \end{bmatrix}$$

You can also find this using row reduction or substitution.