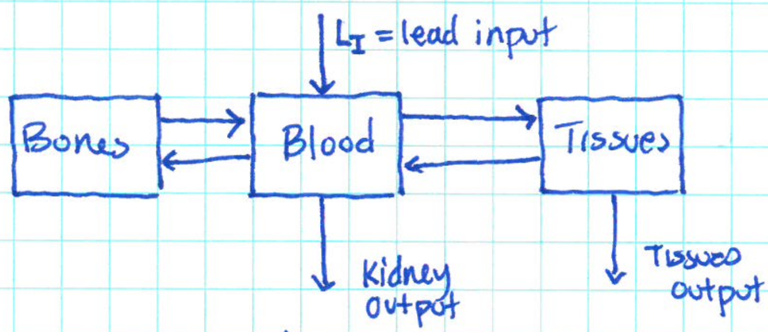
Ex Tracking Lead transport in the body...

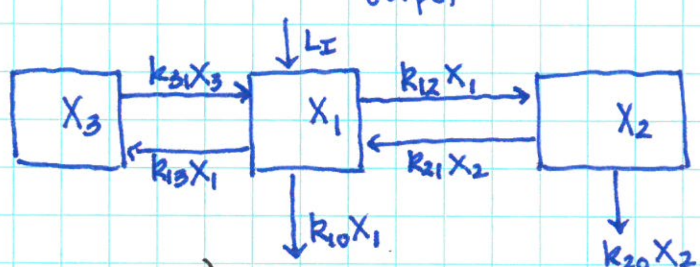
- Lead is found in everyday life: car batteries, water pipes, glassware, ceramics, paint...
- Lead enters the blood stream through food air and water
- Lead leaves the body through the kidneys, and body tissues... sweat, hair, nails.
- Lead is toxic... high levels in the blood and tissues impair motor skills and mental capacity.
- Lead accumulates in the blood bones and tissues.

Draw a Basic Picture of the System



You can make these diagrams up for most systems!

Compartmental Diagrams!



k_{mn} rate of transfer of lead from system X_m to X_n

$X_0 \Rightarrow$ output.

$X_1\ X_2\ X_3$ = amt of lead. (micrograms)

We can write a system of equations for this!

$$\frac{dx_1}{dt} = -k_{10}x_1 - k_{13}x_1 - k_{12}x_1 + k_{31}x_3 + k_{21}x_2 + L_I$$

$$\frac{dx_2}{dt} = -k_{20}x_2 - k_{21}x_2 + k_{12}x_1$$

$$\frac{dx_3}{dt} = -k_{31}x_3 + k_{13}x_1$$

Gather like terms:

$$x_1' = -(k_{10} + k_{13} + k_{12})\,x_1 + k_{21}x_2 + k_{31}x_3 + L_I$$

$$x_2' = k_{12}x_1 - (k_{20} + k_{21})\,x_2$$

$$x_3' = k_{13}x_1 \qquad\qquad\qquad - k_{31}x_3$$

In matrix form:

$$\vec{x}\,' = \begin{bmatrix} -(k_{10}+k_{13}+k_{22}) & k_{21} & k_{31} \\ k_{12} & -(k_{20}+k_{21}) & 0 \\ k_{13} & 0 & -k_{31} \end{bmatrix} \vec{x} + \begin{bmatrix} L_I \\ 0 \\ 0 \end{bmatrix}$$

- so this is of the form $\vec{x}\,' = A\vec{x} + \vec{f}(t)$
- right now if $L_I = 0$ we can solve this eqn using eigenvalue methods
- The more realistic case $L_I = f(t)$ we need another technique...

This is a very complicated system – solve using a computer.

min safe = 5 $\mu g/dL$ $5 \times 50 = 250\ \mu g$

Flint Water = 27 ppb $\cong$ 27 $\mu g/L$ = 6.4 $\mu g/\text{cup}$.

half life of lead: blood = 25 days
tissues = 40 days
bones = 10 years = 3650 days.

Look at the example on SageMath...

Notes. $\vec{x}' = A\vec{x} + \vec{f}(t)$ $\vec{x}(0) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

tells us about the impact of lead if the person started with none in the system.

$\vec{x}' = A\vec{x}$ $\vec{x}(0) = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$

tells us about how long it would take a person to recover if they started with $\vec{x}(0)$ ammounts.

LEAD Example

March 24, 2017

```
## PARAMETERS ##

# Toxin in water
PPB = 5 # Parts per billion measured in water = micrograms per liter
#NOTE
    # average FLINT = 27 (original city study)
    # high FLINT = 158 (original city study)
    # epa standard = 5
    # largest FLINT VA sample = 13,000 (follow up study) - above \
   5000 is considered by EPA to be toxic waste

# Water intake Daily
NumLitersDaily = 1 # Numer of Liters per day drinking water
#NOTE
    # 1 Liter for Children
    # 2 Liters for Adults

# Blood Volume
Blood=1.75 # Blood Volume
#NOTE
    # about 5 Liters for Adults (150 lbs)
    # about 1.75 Liters for Children (50lbs)

# Saftey Standards
mGPerdL=10 #"Safe" Level
# No safe Blood level has been found However testing guidelines say
# NOTE
    # 5 is cause for concern
    # 10 could have neurological effects
    # 40-45 limit before chelation treatment should occur


# Contact through showering
SHOWER = 0 # Set to 1 to consider the effects of a daily shower
```

```
   assume:
a1=var('a1')
a2=var('a2')
a3=var('a3')
NH=matrix(3,1, [a1, a2, a3])

# Solve for NonHomogeneous Constants
# We plug NH into the equation x'=Ax+f...
# (here the derivative of the constant is zero so we always have the\
     following simple system to solve)
CONST=solve([A[0,0]*a1 + A[0,1]*a2 + A[0,2]*a3 + T1==0, A[1,0]*a1 +\
     A[1,1]*a2 + A[1,2]+T2==0, A[2,0]*a1 + A[2,1]*a2 + A[2,2]*a3+T3\
    ==0], a1, a2, a3)
A1=CONST[0][0].rhs()
A2=CONST[0][1].rhs()
A3=CONST[0][2].rhs()
A= matrix(3,1,[A1,A2,A3])

# Apply the initial conditions to the full solution x=x_h+x_nh
# Ax(o)=-f(0)+I, where I contains the initial values
Axo=matrix(3,3,[E1[0,0],E2[0,0],E3[0,0], E1[0,1],E2[0,1],E3[0,1], E1\
    [0,2],E2[0,2],E3[0,2]])
RHSo=matrix(3,1,[-A[0]+I[0], -A[1]+I[1], -A[2]+I[2]])
co=Axo.inverse()*RHSo

## CONSTRUCT FINAL SOLUTION ##
t=var('t')
x1=co[0]*E1[0,0]*e^(L1*t)+co[1]*E2[0,0]*e^(L2*t)+co[2]*E3[0,0]*e^(L3\
    *t)+A[0]
x2=co[0]*E1[0,1]*e^(L1*t)+co[1]*E2[0,1]*e^(L2*t)+co[2]*E3[0,1]*e^(L3\
    *t)+A[1]
x3=co[0]*E1[0,2]*e^(L1*t)+co[1]*E2[0,2]*e^(L2*t)+co[2]*E3[0,2]*e^(L3\
    *t)+A[2]
# NOTE - this works because I have distinct eigenvalues so we just \
    do a simple sum of solutions.
# You would need to be caureful if you have a system with repeated \
    eigenvalues


## PLOT THE SOLUTION ##
f1(t)=x1[0]
f2(t)=x2[0]
f3(t)=x3[0]
f4(t)=SAFE

tend=500
p1=plot(f1(t),(t,0,tend),rgbcolor=hue(0.3),legend_label='Blood')
```

There are many interesting ecological models that use ideas from linear systems.

Ex: Interacting Species: Anteater and Ants.

About the system:

- Anteaters eat ants (duh!)
  more ants available means higher birth rate
- Ants are eaten by anteaters
  more anteaters means higher death rate
- Anteater have some constant death rate
- Ants birth rate is cyclic – less in winter more in summer.

COMPARTMENTAL DIAGRAM

$$\xrightarrow{\text{BIRTH}} \boxed{\begin{matrix} A_1 \\ \text{Anteater} \end{matrix}} \xrightarrow{\text{DEATH}}$$

$$\xrightarrow{\text{BIRTH}} \boxed{\begin{matrix} A_2 \\ \text{Ant} \end{matrix}} \xrightarrow{\text{DEATH}}$$

Basic – now we need formulas!

Assume: Births of anteaters proportional to # of ants.

$\text{BIRTH} \propto A_2$ or $B_{A1} = k_1 A_2$

Death of anteaters constant $\text{DEATH} = C$

Birth of ants is cyclic $\text{BIRTH} = \cos(2\pi t) + 1$

Death of ants proportional to # of anteaters.

$\text{DEATH} \propto A_1$ $D_{A2} = k_2 A_1$

NEW DIAGRAMS

$$\xrightarrow{k_1 A_2} \boxed{A_1} \xrightarrow{C} \qquad \xrightarrow{\cos(2\pi t)+1} \boxed{A_2} \xrightarrow{k_2 A_1}$$

NOTE: On exam you would be given diagrams and formulas...

Write down the equations:

$$\frac{dA_1}{dt} = k_1 A_2 - C$$

$$\frac{dA_2}{dt} = -k_2 A_1 + \cos(2\pi t) + 1$$

in MATRIX FORM.

$$\vec{A}' = \begin{bmatrix} 0 & k_1 \\ -k_2 & 0 \end{bmatrix} \vec{A} + \begin{bmatrix} -C \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \cos(2\pi t)$$

Now — from experiment or observation we would find the constants.

let $k_1 = 1/4$
$k_2 = 1/4$
$C = 2$

Let's solve this system and make some conclusions... easier system... solve by hand.

lets make up some numbers for our constants ...

$$k_1 = 1/4 \qquad k_2 = 1/4 \quad \text{and} \quad C = 2$$

- in real life we would find these from observation or experiment!

$$\vec{A}' = \begin{bmatrix} 0 & 1/4 \\ -1/4 & 0 \end{bmatrix} \vec{A} + \begin{bmatrix} -2 \\ \cos(2\pi t) + 1 \end{bmatrix}.$$

$$X_H = c_1 \begin{bmatrix} \cos(t/4) \\ -\sin(t/4) \end{bmatrix} + c_2 \begin{bmatrix} \sin(t/4) \\ \cos(t/4) \end{bmatrix}$$

Make sure you can find homogeneous solns using e-val method.

Now find X_p ... $f(t) = \begin{bmatrix} -2 \\ \cos(2\pi t) + 1 \end{bmatrix}$

Guess: $$X_p = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \sin(2\pi t) + \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \cos(2\pi t)$$

Find all the pieces and plug into ODE ...

$$X_p' = \begin{bmatrix} 2\pi b_1 \\ 2\pi b_2 \end{bmatrix} \cos(2\pi t) + \begin{bmatrix} -2\pi c_1 \\ -2\pi c_2 \end{bmatrix} \sin(2\pi t)$$

$$AX_p = A \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} + A \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \sin(2\pi t) + A \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \cos(2\pi t)$$

$$= \begin{bmatrix} 1/4\, a_2 \\ -1/4\, a_1 \end{bmatrix} + \begin{bmatrix} 1/4\, b_2 \\ -1/4\, b_1 \end{bmatrix} \sin(2\pi t) + \begin{bmatrix} 1/4\, c_2 \\ -1/4\, c_1 \end{bmatrix} \cos(2\pi t)$$

$$c_2 = 0 \quad b_1 = 0 \quad c_1 = \frac{1}{64\pi^2 - 1} \quad b_2 = \frac{-8\pi}{64\pi^2 - 1}$$

$$a_1 = 4 \qquad a_2 = 8$$