- ~~Hand IN 24~~
  ~~(and 23)~~
- ~~25 due Wed~~ • ~~26 due Friday~~

Lecture 29 Diff Eq.

Nonlinear Systems and Phenomena — now that we have eigenvalue solns we can investigate nonlinear systems!

General Form

$$\frac{dx}{dt} = F(x,y)$$
$$\frac{dy}{dt} = G(x,y)$$

F and G not necessarly linear.

This is called and autonomous system ... the absence of t on the right hand side makes it easier to analyze.

Using P-plane we can plot phase planes for these systems. Solutions are trajectories in the phase plane.

Recall FIXED POINTS / CRITICAL POINTS — important things happen when there is no change in the system.

$$F(x^*, y^*) = 0 \text{ and } G(x^*, y^*) = 0.$$

$x(t) = x^*$ and $y(t) = y^*$ are equlibrium solutions for the system.

Ex Find all the critical points of the system:

$$\left.\begin{aligned} x' &= 14x - 2x^2 - yx \\ y' &= 16y - 2y^2 - yx \end{aligned}\right\}$$

we must solve for all possible ways the RHS = 0.

* imagine that x(t) gives the # of rabbits at time t and y(t) gives the # of squirrels at time t.

Set RHS = 0 SOLVE.

$14x - 2x^2 - xy = 0 \Rightarrow x(14 - 2x - y) = 0$

$16y - 2y^2 - xy = 0 \qquad y(16 - 2y - x) = 0$

TWO CASES TOP: $x = 0$ or $14 - 2x - y = 0$

TWO CASES BOTTOM: $y = 0$ or $16 - 2y - x = 0$

FOUR POSSIBLE CRITICAL POINTS.

$x = 0$ and $y = 0$ - - - - - - → $(0,0)$

or $16 - 2y = 0$ - - - - - → $(0,8)$

$14 - 2x - y = 0$ and $y = 0$ - - - - - - → $(7,0)$

$16 - 2y - x = 0$

solve for $14 - 2x - y = 0 \qquad y = -2x + 14$

$16 - 2y - x = 0$

$16 + 4x - 28 - x = 0$

$3x - 12 = 0 \qquad 3x = 12 \quad x = 4$

$y = -8 + 14 = 6$

$(4,6)$

Critical Points $(0,0)(0,8)(7,0)(4,6)$

What do these mean? $(0,0)$ no squirrels no rabbits

$(0,8)$ 8 squirrels no rabbits

$(7,0)$ no squirrels 7 rabbits

$(4,6)$ 6 squirrels and 4 rabits → the only chance for coexistence!

Lets plot this system on Pplane and see what is happening. ON COMPUTER.

YOU TRY:

$x' = x - y$

$y' = 1 - x^2$

Is the soln likely to hit one of these points?

Our trajectories seem to go toward $(4,6)$

GOOD NEWS FOR THE SQUIRRELS AND RABBITS!

Wouldn't it be nice if we could just look at our system of equations and know about the behavior near critical points?

This is our GOAL → we will start simple and build up to this idea!

Consider the system:

$$\frac{dx}{dt} = -x$$

$$\frac{dy}{dt} = -ky$$ where k is a non zero constant.

- FIXED POINTS $x=0$ $y=0$
  only one critical point $(0,0)$.

* Now what is hapening at the fixed point for different values of k?

STABLE – solutions go toward the fixed point
the system wants to head toward the fixed point (or stay near)

UNSTABLE – solutions go away from the fixed point
the system does not want to head toward the fixed point. (or move away)

We can solve this system: $x(t) = x_0 e^{-t}$ $\quad y(t) = y_0 e^{-kt}$

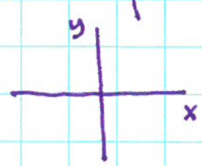
rewrite $y(t) = y_0 \left(e^{-t}\right)^k \underset{\text{from } x(t)}{=} y_0 \left(x/x_0\right)^k = cx^k$

$\frac{y_0}{x_0^k}$ just a constant

NOTE. eigenvalues $\lambda = -1, -k$

so our trajectories in the phase plane are $y = cx^k$.

y is related to x!

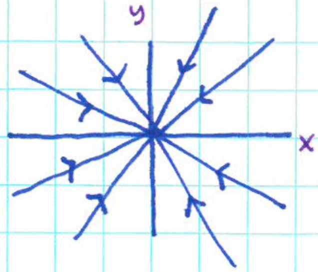


CASE 1 $k > 0$

A. then if $k = 1$ then $y = cx$

trajectories are straight lines.

$\lambda = -1, -1$
equal eigenvalues.



PROPER NODE

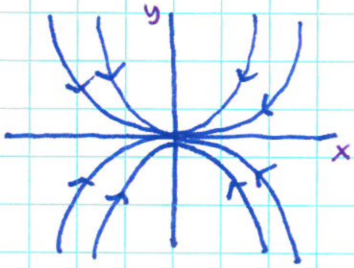
Both x and y decrease directly to $(0,0)$

$(0,0)$ stable Fixed Point.

B. if $k > 1$ then $y = cx^n$

trajectories are parabolas. or polynomials w/ repeated root

THESE ARE ALSO CALLED SINKS!



IMPROPER NODE

y approaches origin faster than x.

$\lambda = -1, -a^2$
$a > 1$

— decreases faster in y

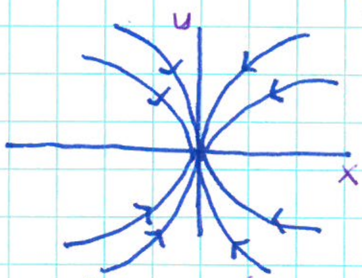
$(0,0)$ stable Fixed Point.

different but same sign.

C. if $k < 1$ [but > 0] $y = cx^{1/n}$

trajectories "hyperbolas"

$\lambda = -1, -a^2$
$a < 1$



IMPROPER NODE

x approaches origin faster than y.

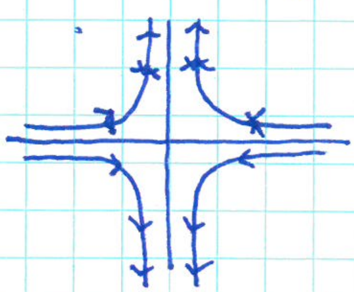
decreases faster in x.

$(0,0)$ stable Fixed Point.

CASE 2 $k < 0$ then $y = cx^{-n}$

trajectories are hyperbolas.
cannot cross $x = 0$.

$\lambda = -1, a^2$
different sign. eigenvalues.



SADDLE POINT

one solution grows while the other decays.

decreases in x
increases in y.

$(0,0)$ Unstable Fixed Point.

NOTE: If our original system was $\frac{dx}{dt} = x$, $\frac{dy}{dt} = ky$

eigenvalues $\lambda = 1, k$

It all has to do with direction (+/-) of the eigenvalues!

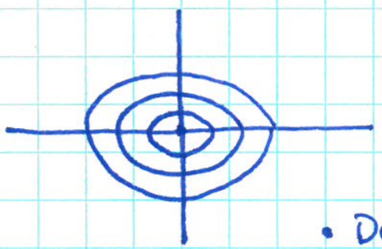
the trajectories are the same but the arrows are flipped.

$x(t) = x_0 e^{t}$ $y = y_0 e^{kt}$ growing exponentials.

We can also have <u>elliptical</u> or <u>spiral</u> trajectories!

<u>Ex</u> $\frac{dx}{dt} = y$, $\frac{dy}{dt} = -\omega^2 x$ $\Rightarrow$ $x(t) = A\cos\omega t + B\sin\omega t$, $y(t) = -A\omega\sin\omega t + B\omega\cos\omega t$

(0,0) is a FIXED POINT



ELLIPTICAL TRAJECTORIES. pure imaginary solns.

- Depending on the initial disturbance the soln oscillates about the fixed point.

We would call this fixed point ★ ~~asymptotically~~ stable

$\lambda = $ imag

- a solution that starts near the fixed point remains close ... it does not move AWAY..

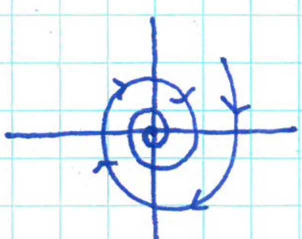
<u>Ex</u> $\frac{dx}{dt} = y$, $\frac{dy}{dt} = -2x - 2y$

$x(t) = e^{-t}(A\cos t + B\sin t)$
$y(t) = e^{-t}((B-A)\cos t - (A+B)\sin t)$

$\lambda =$ complex.

(0,0) is a FIXED POINT

A spiral sink.



STABLE soln spirals toward (0,0).

SPIRAL TRAJECTORIES complex solutions.

- depending on the real part this can be stable or unstable

NOTE: The only thing changing from system to system.

Type of eigenvalues !

$$\frac{dx}{dt} = -x \qquad \frac{dx}{dt} = -ky \Rightarrow \begin{vmatrix} -1 & 0 \\ 0 & -k \end{vmatrix} \Rightarrow (-1-\lambda)(-k-\lambda)$$

$\lambda = -1, -k$

Real Eigenvalues.

- EQUAL: $k=1$
  proper node.
  $\lambda = -1, -1$
  $\lambda = -1 \Rightarrow$ decaying
  STABLE.

- NOT EQUAL SAME SIGN $\lambda = -1$ $\lambda = -\frac{1}{2}$
  Improper Node
  Decaying e-vals STABLE.

- OPPOSITE SIGNS $\lambda = -1$ $\lambda = 1$
  SADDLE POINT.
  UNSTABLE
  one soln always grows.

What is the difference...

$$\frac{dx}{dt} = x \qquad \frac{dx}{dt} = ky \Rightarrow \underbrace{\lambda = 1 \quad \lambda = k}$$

now we get growing solns,

FIXED POINTS BECOME UNSTABLE.

- our eigenvalues are telling us everything we need to know !

IMAGINARY $\rightarrow$ ELIPTICAL ORBITS.

COMPLEX $\rightarrow$ SPIRAL ORBITS.

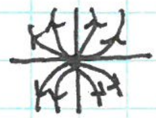
TWO POSITIVE λ.

$\lambda_1 = \lambda_2$

proper node
unstable

$\lambda_1 \neq \lambda_2$

improper node
unstable



$e^{\lambda t}$ solns are growing

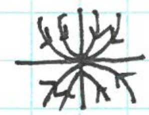
TWO NEGATIVE λ

$\lambda_1 = \lambda_2$

proper node
stable

$\lambda_1 \neq \lambda_2$

improper node
stable

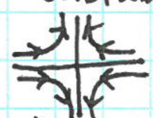


$e^{\lambda t}$ solns are decaying

OPPOSITE SIGNS

$\lambda_1 > 0 \quad \lambda_2 < 0$

saddle point
unstable



one $e^{\lambda t}$ grows
the other decays.

[direction of flow depends on which soln is growing]

IMAGINARY

$\lambda = \pm ai$

elliptical trajectories

CENTERS.
"stable!"

but more complicated in the full nonlinear system.

COMPLEX

$\lambda = b \pm ai$

spiral trajectories.

$Re(\lambda) > 0$

unstable
e^{bt} growing

$Re(\lambda) < 0$

stable
e^{bt} decaying.