Lecture 30 ~~29~~ Diff Eq.

- Hand in HW 24
- HW 25 Due Friday
- HW 26 Due Monday.

I am going to try to get the final Exam to you Friday 4pm. Due Monday at 4pm [Three Days]

LAST TIME → Fixed Points and Stability

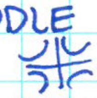
PROPER NODE when $\lambda_1 = \lambda_2$

$\lambda_1 \lambda_2 < 0$ STABLE
$\lambda_1 \lambda_2 > 0$ UNSTABLE

IMPROPER NODE when $\lambda_1 \neq \lambda_2$ but same sign.

$\lambda_1 \lambda_2 < 0$ STABLE
$\lambda_1 \lambda_2 > 0$ UNSTABLE

SADDLE POINT when $\lambda_1 \neq \lambda_2$ and opposite signs.



ALWAYS UNSTABLE.

ELLIPTICAL ORBITS (limit cycles) when λ_1 and λ_2 pure imaginary

ASYMPTOTICALLY STABLE

SPIRALS when λ_1 and λ_2 complex.

$Re\{\lambda_1, \lambda_2\} < 0$ STABLE
$Re\{\lambda_1, \lambda_2\} > 0$ UNSTABLE.

LETS APPLY THIS IDEA TO A NONLINEAR SYSTEM:

Assume we are given ... $\frac{dx}{dt} = f(x,y) \quad \frac{dy}{dt} = g(x,y)$

and that this system has a critical point at (x_0, y_0).

- We want to zoom into the point (x_0, y_0) and determine the type and stability.
- If we zoom in on a curve ... eventually get a line!
- Zoom in on our fixed point we LINEARIZE THE SYSTEM NEAR THE FIXED POINT.

Taylors Formula (function of two variables)

We can approximate $f(x,y)$ near (x_0, y_0) by

$$f(x+x_0, y+y_0) = f(x_0,y_0) + f_x(x_0,y_0)x + f_y(x_0,y_0)y + \text{HOT.}$$

HOT → The Higher Order Terms, the nonlinear parts of the approximation

Plug this into the ODE

$$\frac{dx}{dt} = f(x_0,y_0) + f_x(x_0,y_0)x + f_y(x_0,y_0)y$$

$$\frac{dy}{dt} = g(x_0 y_0) + g_x(x_0 y_0)x + g_y(x_0,y_0)y$$

BUT if (x_0,y_0) is a critical point then $f(x_0,y_0)=0$ and $g(x_0,y_0)=0$.

SO OUR APPROXIMATE LINEAR SYSTEM IS:

$$\frac{dx}{dt} = f_x(x_0 y_0)\,x + f_y(x_0 y_0)\,y$$

$$\frac{dy}{dt} = g_x(x_0 y_0)\,x + g_y(x_0 y_0)\,y$$

in MATRIX FORM

$$\vec{x}\,' = \begin{vmatrix} f_x(x_0 y_0) & f_y(x_0 y_0) \\ g_x(x_0 y_0) & g_y(x_0 y_0) \end{vmatrix} \vec{x}$$

↳ does this look familiar?

$$\text{JACOBIAN} = \frac{\partial(f\ g)}{\partial(x\ y)} = \begin{vmatrix} f_x & f_y \\ g_x & g_y \end{vmatrix}$$

So our analysis of fixed points in nonlinear systems boils down to finding eigenvalues of the Jacobian Matrix evaluated at $(x_0\, y_0)$!

Ex $\quad \vec{x}' = 3x - x^2 - xy$
$\quad\quad y' = y + y^2 - 3xy$

1. Find fixed points: $\quad 3x - x^2 - xy = 0$
$\quad\quad y + y^2 - 3xy = 0$

$x(3 - x - y) = 0$ $\quad$ Four fixed Points
$y(1 + y - 3x) = 0$

$x = 0$ then $\quad y = 0 \quad \longrightarrow \quad y = 0$
$\quad\quad 1 + y - 3\cancel{x}^{0} = 0 \quad\quad y = -1$

$y = 0$ then $\quad 3 - x - \cancel{y}^{0} = 0 \longrightarrow x = 3$

$3 - x - y = 0$
$1 + y - 3x = 0$ $\rightarrow$ $4 - 4x = 0$ $\rightarrow$ $x = 1$
$\quad y = 3 - x = 2$

$(0,0)\ (0,-1)\ (3,0)\ (1,2)$ ✓

2. Now we can analyze the stability of these fixed points.

A. JACOBIAN $= \begin{vmatrix} f_x & f_y \\ g_x & g_y \end{vmatrix} = \begin{vmatrix} 3-2x-y & -x \\ -3y & 1+2y-3x \end{vmatrix}$

B. Choose a point ... $(0,0)$ plug into JACOBIAN

$$\begin{vmatrix} 3 & 0 \\ 0 & 1 \end{vmatrix} = J\Big|_{(0,0)}$$

C. Find eigenvalues of this matrix.

$$\begin{vmatrix} (3-\lambda) & 0 \\ 0 & (1-\lambda) \end{vmatrix} = (3-\lambda)(1-\lambda) = 0 \quad \lambda = 3, 1$$

D Classify the point.

(0,0) is an IMPROPER NODE
with positive real eigenvalues
It is UNSTABLE.

You Try the other ones !!

$$J\Big|_{(0,-1)} = \begin{vmatrix} 4 & 0 \\ 3 & -1 \end{vmatrix} \qquad \begin{vmatrix} 4-\lambda & 0 \\ 3 & -1-\lambda \end{vmatrix} = (4-\lambda)(-1-\lambda) = 0$$

$\lambda = 4$ and $\lambda = 1$

~~IMPROPER NODE~~ SADDLE
UNSTABLE

$$J\Big|_{(3,0)} = \begin{vmatrix} -3 & -3 \\ 0 & -8 \end{vmatrix} \qquad \begin{vmatrix} -3-\lambda & -3 \\ 0 & -8-\lambda \end{vmatrix} = (-3-\lambda)(-8-\lambda)$$

$\lambda = -3 \quad \lambda = -8$

IMPROPER NODE
STABLE

$$J\Big|_{(1,2)} = \begin{vmatrix} -1 & -1 \\ -6 & +2 \end{vmatrix} \qquad \begin{vmatrix} -1-\lambda & -1 \\ -6 & 2-\lambda \end{vmatrix} = (-1-\lambda)(2-\lambda) - 6$$

$$= -2 - \lambda + \lambda^2 - 6 = \lambda^2 - \lambda - 8$$

$$\lambda = \frac{1 \pm \sqrt{1+32}}{2} = \frac{1}{2} \pm \frac{\sqrt{\cancel{31}\ 33}}{2} \quad \sim 2.8$$

SADDLE
UNSTABLE

consider the associated eigenvectors for each case

$(0,0)$ $\quad J\Big|_{00} = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}$

$\lambda = 3, 1$

$\lambda = 3 \quad \begin{bmatrix} 0 & 0 \\ 0 & -2 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \emptyset \qquad V_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$\lambda = 1 \quad \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \emptyset \qquad V_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$(0,-1)$ $\quad J\Big|_{0-1} = \begin{bmatrix} 4 & 0 \\ 3 & -1 \end{bmatrix}$

$\lambda = 4, -1$

$\lambda = 4 \quad \begin{bmatrix} 0 & 0 \\ 3 & -5 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \emptyset \qquad \begin{bmatrix} 1 \\ 3/5 \end{bmatrix}$

$\lambda = -1 \quad \begin{bmatrix} 5 & 0 \\ 3 & 0 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \emptyset \qquad \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$(3,0)$ $\quad J\Big|_{30} = \begin{bmatrix} -3 & -3 \\ 0 & -8 \end{bmatrix}$

$\lambda = -3, -8$

$\lambda = -3 \quad \begin{bmatrix} 0 & -3 \\ 0 & -5 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \emptyset \qquad V_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

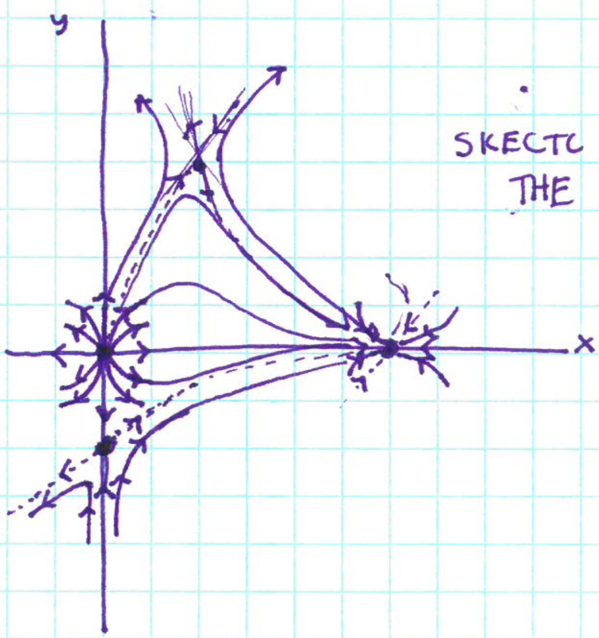
$\lambda = -8 \quad \begin{bmatrix} 5 & -3 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \emptyset \qquad V_2 \begin{bmatrix} 1 \\ 5/3 \end{bmatrix}$

$(1,2)$ $\quad J\Big|_{12} = \begin{bmatrix} -1 & -1 \\ -6 & 2 \end{bmatrix}$

$\lambda = 1/2 \pm \sqrt{33}/2$

$V_1 = \begin{bmatrix} 1 \\ 1.372 \end{bmatrix} \quad \lambda_1 < 0 \sim -2.37$

$V_2 = \begin{bmatrix} 1 \\ -4.372 \end{bmatrix} \quad \lambda_2 > 0 \sim 3.37$



SKECTC THE Pplane.

LOTS OF APPLICATIONS !!

EX SIR MODEL OF INFECTIOUS DISEASE.

Three categories of people:

SUSCEPTIBLE → INFECTIOUS → RECOVERED

1. Assume the total population is constant
2. Infectious people run into susceptible people and transmit the disease at a rate of β
3. Only a percentage of the infectious people recover but then they are no longer susceptible.

$$\frac{dS}{dt} = -\beta S I$$

• law of mass action chem.

$SI \rightarrow$ models the probability of a susceptible person running into an infected person.

$$\frac{dI}{dt} = \underbrace{\beta S I}_{\text{newly infected}} - \underbrace{\alpha I}_{\text{recovering people or people no longer infected}}$$

$\beta \rightarrow$ the chance that the S person becomes infected.

α = recovery rate.

$$\frac{dR}{dt} = (\alpha - d) I$$

$(\alpha - d) \rightarrow$ the % of recoverd people who remain alive.

Aside:

NOTES: If N is the total # of people then

βN = # of newly infected per unit time

α = fraction of infectives that leave per unit time

so $1/\alpha$ = average time of recovery

Imagine a Visitor Comes to Campus with a Disease.

$$\begin{cases} N = \# \text{ of People } = 2500 \\ \beta N = 5 \quad \text{so} \quad \beta = 5/2500 = .002 \quad \text{rate of transmission} \\ \alpha = 1/5 \quad \text{takes 5 days to recover} \\ d = 0 \quad \text{a non-lethal disease.} \end{cases}$$

OUR EQNS BECOME

$$S' = -.002\,SI$$
$$I' = .002\,SI - .2I$$
$$R' = .2I$$

1. What are my fixed points?

$SI = 0$
$I(.002S - .2) = 0$
$I = 0$

- only real fixed point $I = 0$ ✓ with S and R arbitrary.
- Does this make sense?

$I = 0 \Rightarrow$ zero infected so either $S = 2500$ or $R = 2500$.

so Technically $(2500, 0, 0)$ or $(0, 0, 2500)$ are my fixed points.

2. Jacobian: $\dfrac{\partial(f, g, h)}{\partial(S, I, R)} = \begin{vmatrix} f_S & f_I & f_R \\ g_S & g_I & g_R \\ h_S & h_I & h_R \end{vmatrix}$

$$J = \begin{vmatrix} -.002I & -.002S & 0 \\ .002I & .002S - .2 & 0 \\ 0 & .2 & 0 \end{vmatrix}$$

3. Fixed Points Analysis

$$J\Big|_{\substack{(2500,0,0)\\(S,R,I)}} = \begin{vmatrix} 0 & 5 & 0 \\ 0 & 4.8 & 0 \\ 0 & .2 & 0 \end{vmatrix} \qquad \begin{vmatrix} -\lambda & 5 & 0 \\ 0 & 4.8-\lambda & 0 \\ 0 & .2 & -\lambda \end{vmatrix} = -\lambda(-\lambda(4.8-\lambda))$$

$$= \lambda^2(4.8-\lambda) = 0$$

$\lambda = 0, 4.8$ no negative λ's this is UNSTABLE.

$$J\Big|_{(0,0,2500)} = \begin{vmatrix} 0 & 0 & 0 \\ 0 & -.2 & 0 \\ 0 & .2 & 0 \end{vmatrix} \quad \begin{vmatrix} -\lambda & 0 & 0 \\ 0 & -.2-\lambda & 0 \\ 0 & .2 & -\lambda \end{vmatrix} = -\lambda(-\lambda(-.2-\lambda))$$

$$= \lambda^2(-.2-\lambda) = 0$$

$\lambda = 0, -.2$ negative eigenvalues
this is STABLE.

Just looking @ non-zero e-vals.

$\lambda = 4.8$ and $\lambda = -.2$

$\lambda = 4.8$: even one sick person and immediately others get sick – this happens very quickly $\sim e^{4.8t}$

$\lambda = -.2$: after people are sick eventually they all recover but it takes a while $\sim e^{-.2t}$

The magnitude of the eigenvalue tells us about the system too!

Can't draw a simple phase plane because we have a 3-D system!

Interesting Idea... BIFURCATIONS – if the disease became more or less transmitable, could that change the stability of our fixed points?