Differential Equations – Nonlinear Systems and Applications

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Species Interaction Example

Today we will investigate the interaction of two species that both have logistic growth. Each of the species is negatively effected by the interaction. We could imagine this as two species of wild cat, each of which hunts the other. Here are the equations that model this system:

$$x' = 14\left(1 - \frac{x}{7}\right)x - xy$$
$$y' = 16\left(1 - \frac{y}{8}\right)y - cxy$$

Here the numbers 14 and 16 represent the maximum reproductive rate of each species. This means that species y has a slightly higher reproduction rate than species x. The numbers 7 and 8 represent the ecological carrying capacity, meaning that the ecosystem can sustain 7 of species x and 8 of species y before the population exceeds the limited resources. The interaction term is the term we will investigate! Here the parameter c represents what happens when x and y interact. We can see that each interaction decreases x but the effect on species y depends on the value of c. We will do a fixed point analysis of this system. Here we will assume that $c > 0$

1. First find the fixed points of the system. Discuss what they mean. What are the possibilities for the value of c? In other words, when do we see a realistic fourth critical point?
2. Now analyze the fixed points using the Jacobian and Eigenvalues:
   - The first three should be straightforward. For the fourth, more complicated, fixed point use SageMath to help you find solutions.
   - Is the stability of the fixed point effected by the value of c?
   - If so, what conditions are required to make the point stable vs. unstable?
   - What does this mean in terms of the ecological system?
3. Plot a few Phase Planes to confirm or check your results.
4. Describe, overall, what you learned about the ecological system from this fixed point analysis.

A Damped Nonlinear Pendulum

The following system of equations models a pendulum or swing that has some damping.

$$x' = y$$
$$y' = -\sin(x) - cy$$

Here we are assuming a frequency of one and the term $c > 0$ represents the amount of damping in the system. The variables x and y represent the parametric representation of the pendulums position over time where x represents the pendulums angle away from rest and y represents the pendulums angular velocity. In other words, if $x = 0$ the the pendulum is hanging straight down, if $x = \pi$ the pendulum is sticking straight up, if $y = 0$ then the pendulum is at rest.

1. First find the fixed points of the system (there will be lots!). Discuss what they mean. Do the fixed points depend at all on c? Does this make sense?
2. Now analyze the fixed points using the Jacobian and Eigenvalues:
   - Here you will be able to classify your fixed points into two distinct cases: even and odd.
   - Is the stability or type of the fixed point effected by the value of c?
   - What do these results mean in terms of the pendulum system being modeled?
3. Plot a few Phase Planes to confirm or check your results. You could try using $c = \frac{1}{4}$ to get nice looking plots.
4. Describe, overall, what you learned about the pendulum system.