Eulers Method 2.4 in the book

GOAL – find a systematic way to approximate an ODE.

We will think about this in terms of a first order problem:

$$y' = f(x,y).$$

could easily generalize to first order systems

Ex $y' = xy \quad y(0) = 1$

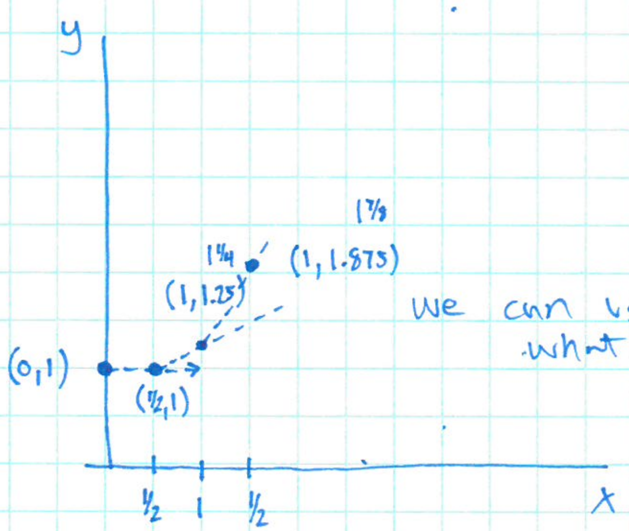
this one we can solve:

$\ln(y) = \frac{1}{2}x^2 + c \qquad y = Ae^{x^2/2}$

$y(0) = A = 1$

$$\boxed{y = e^{x^2/2}}$$

BUT we will pretent, for now, that we can't solve this!



What do we know?

$y(0)=1$ point

and $y' = xy$ slope.

We can use what we know about slope to "shoot" for the next soln.

Q – are we under or over?
– how might we talk about error?
– does ~~our~~ error get better or worse the further we go?

The Method...

How did we get our approximations?

$$y_1 = \underset{\substack{\uparrow \\ \text{starting point.}}}{y(0)} + \underset{\substack{\uparrow \\ \text{how far we step}}}{\tfrac{1}{2}}\, \underset{\substack{\uparrow \\ \text{slope}}}{f(y(0), x(0))}$$

This is <u>EULER'S METHOD</u>!

☆ The Method.

given $\dfrac{dy}{dx} = f(x,y)$ $\quad y(x_0) = y_0$

we write the iterative formula:

$$y_{n+1} = y_n + h \cdot f(x_n, y_n)$$

where h is our step size.

<u>Ex</u> $\quad y' = xy \quad y(0) = 1$

If $h = .5$ then:

Iterative Formula:

$$y_{n+1} = y_n + \tfrac{1}{2}(x_n y_n)$$

x-values	y-values
0	1
.5	$= 1 + \frac{1}{2}(0)(1) = 1$
1	$= 1 + \frac{1}{2}(\frac{1}{2})(1) = 1.250$
1.5	$= 1.250 + \frac{1}{2}(1)(1.250) = 1.875$
2	$= 1.875 + \frac{1}{2}(1.5)(1.875) = 3.28125 \sim 3.281$
2.5	$= 3.281 + \frac{1}{2}(2)(3.281) = 6.562$
3	$= 6.562 + \frac{1}{2}(2.5)(6.652) = 14.765$

what about rounding?

"natural" round down

this increases error! because method is under.

round up

this decreases error because method is under.

	Real Values	ERROR
0	1	0
.5	1.133	.1331
1	1.649	.3987
1.5	3.082	1.2052
2	7.3891	4.1081
2.5	22.760	16.198
3	90.017	75.253

gets pretty big the farther we go!

Rounding errors are tricky!

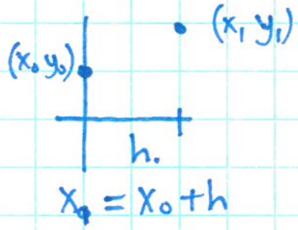
- sometimes rounding helps... hard to know when.
- method accuracy increases as $h \to 0$ but rounding becomes a bigger issue as h gets smaller

Default Matlab Precision: Double 15–17 digits.
(Single 7–8 digits)

We can more generally derive these things using TAYLOR EXPANSIONS.

expand $f(x)$ near $x=a$

$$f(x) = f(a) + f'(a)(x-a) + f''(a)\frac{(x-a)^2}{2} + \dots$$



we just need to approximate $y_1 = f(x_1)$ using what we know at x_0.

expand $f(x_1)$ near x_0

$$f(x_1) = f(x_0) + f'(x_0)(x_1 - x_0) + f''(x_0)\frac{(x_1-x_0)^2}{2} + \dots$$

but $x_1 - x_0 = h$

$$f(x_1) = f(x_0) + f'(x_0)(h) + f''(x_0)\frac{h^2}{2} + \dots$$

solve for this ... first order first derivative.

$$\frac{f(x_1) - f(x_0)}{h} = f'(x_0) + f''(x_0)\frac{h}{2} + \dots$$

as long as h is small these terms are negligible

so $f'(x_0) = \dfrac{f(x_1) - f(x_0)}{h}$

$y' = f(x, y)$
this is just $f(x_0, y_0)$

so $f(x_1) = f(x_0) + h\, f(x_0, y_0)$

in general $f(x_{n+1}) = y_{n+1} = y_n + h f(x_n, y_n)$. Euler Method.

USING the Method.

1. Consider the ode ... think about error, curvature and step size h.

2. Write down the iterative formula for the Given ODE

$$y_{n+1} = y_n + h\underbrace{f(x_n\, y_n)}_{\text{this part changes.}}$$

3. Iterate either by hand or using freemat / Matlab.

NOTE... you can only exactly know the error if you can solve the ODE.