Using the Method.

1. Consider the ode ... think about error, curvature and step size h. (and existence/uniqueness)

2. Write down the iterative formula for the Given ODE

$$y_{n+1} = y_n + h\underbrace{f(x_n\, y_n)}_{\text{this part changes.}}$$

$y' = f(x,y)$

3. Iterate either by hand or using freemat / Matlab.

NOTE... you can only exactly know the error if you can solve the ODE.

$$\text{Error} = |\text{Exact} - \text{Approx}|$$

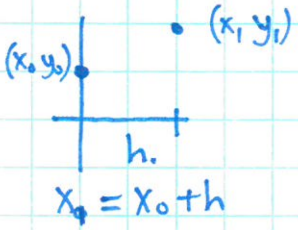
Rounding errors are tricky! — our answers will have less rounding...
- sometimes rounding helps... hard to know when.
- method accuracy increases as $h \to 0$ but rounding becomes a bigger issue as $h$ gets smaller

Default Matlab Precision: Double 15–17 digits.
(Single 7–8 digits)

We can more generally derive these things using **TAYLOR EXPANSIONS**.

expand $f(x)$ near $x=a$

$$f(x) = f(a) + f'(a)(x-a) + f''(a)\frac{(x-a)^2}{2} + \dots$$




we just need to approximate $y_1 = f(x_1)$ using what we know at $x_0$.

expand $f(x_1)$ near $x_0$

$$f(x_1) = f(x_0) + f'(x_0)(x_1 - x_0) + f''(x_0)\frac{(x_1-x_0)^2}{2} + \dots$$

but $x_1 - x_0 = h$

$$f(x_1) = f(x_0) + f'(x_0)(h) + f''(x_0)\frac{h^2}{2} + \dots$$

solve for this... first order first derivative.

$$\frac{f(x_1) - f(x_0)}{h} = f'(x_0) + f''(x_0)\frac{h}{2} + \dots$$

as long as $h$ is small these terms are negligible

so $f'(x_0) = \dfrac{f(x_1) - f(x_0)}{h}$

$y' = f(x,y)$
this is just
$f(x_0, y_0)$

so $f(x_1) = f(x_0) + h\, f(x_0, y_0)$

in general $f(x_{n+1}) = y_{n+1} = y_n + h f(x_n, y_n)$. Euler Method.

Lets think about the method error:

$$\frac{f(x_1)-f(x_0)}{h} = f'(x_0) + f''(x_0)\, h/2 + \cdots$$

so really $$f'(x_0) = \frac{f(x_1)-f(x_0)}{h} \underbrace{- f''(x_0)\, h/2 + \cdots}_{\text{the HOT tell us about the error.}}$$

we assume that $h$ is small
so $h^2$ is ... and $h^3$ is ... REALLY SMALL.

We could approximate our method error by thinking about

$$-f''(x_0)\frac{h}{2} = -\frac{f''(x_0)}{2}\, h$$

$f''(x_0) > 0$ then concave up $\cup$ Error $< 0$ under estimate

$f''(x_0) < 0$ then concave down $\cap$ Error $> 0$ over estimate.

This matches our instinct! We also see that magnitude of the error has something to do with curvature.

What about step size if we half our step size.

$$h \longrightarrow h/2$$

we approximately halve our error.

This is a <u>FIRST ORDER</u> METHOD
double # of steps ~ half the error.

NOTE: Higher order Methods are really cool:
SECOND ORDER ~ $h^2$ so double # steps ~ quarter the error.

IN PRACTICE... we get a sense of the error by comparing
our answer → $h$, $h/2$, $h/4$.

$$\text{Error} \cong \left| y_h - y_{h/2} \right|$$

ERROR...

PROBLEMS 25 and 29.

$v' = f(v,t) =$

25 $32 - 1.6v$ could calculate $v'' = -1.6v' \leq 0$ expect over.

we are falling so v is increasing and $v' > 0$

| h | y(1) | "Error" |
|---|---|---|
| .1 | 16.5020 | X |
| .05 | 16.2261 | 16.5020 − 16.2261 = 0.2759 |
| .025 | 16.0927 | = 0.1334 |
| .0125 | 16.0270 | = 0.0657 |
| .00625 | 15.9945 | = 0.0325 |

$\frac{.2759}{.1334} = 2.0682$

$\frac{.1334}{.0657} = 2.0304$

$\frac{.0657}{.0325} = 2.0215$

this shows that the approximate error is decreasing

this shows our method is order 1

$h \to h/2$

error $\to$ error/2