Lecture Differential Eqns.

- Hand Back exams
- HW 8 Due Friday.

Ch. 3 Linear Eqns of Higher Order...

Some ideas that we will use for the rest of class:

1. Homogeneous vs. Nonhomogeneous
2. Principle of Superposition
3. The Wronskian and Linear Independence
4. Idea of a General Soln.

We will focus on Second Order Linear Eqns:

$$e^x y'' + \cos x \, y' + (1+\sqrt{x})y = \tan^{-1} x \qquad \text{YES}$$

$$y'' + 3(y')^2 + 4y^3 = 0 \qquad \text{NO}$$

NOTE: It is possible to reduce an order n equation into n order one equations and solve as a system!

The GENERAL FORM:

$$A(x)y'' + B(x)y' + C(x)y = F(x) \qquad y(a)=b_0 \quad y'(a)=b_1$$

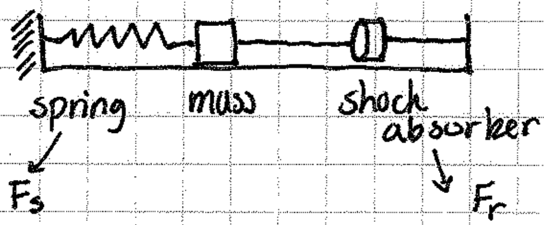
or $$y'' + P(x)y' + q(x)y = r(x)$$
(sometimes we divide by $A(x)$)

As long as $A, B, C,$ and F are continuous on some open interval containing the point a solutions exist and are unique.

Why do we care about higher order eqns?

In the real world we have acceleration!

Ex Mass-Spring EQN



We assume the force of the spring is proportional to the displacement:

$$F_s = -kx$$ $x=0$ the spring/mass is at equilibrium.

the force of shock absorber is proportional to velocity

$$F_r = -cv$$ pushing opposite to the motion.

Newtons Law $\sum F = ma = mx''$

$$mx'' = F_s + F_r = -kx - cx'$$

so $mx'' + kx + cx' = 0$
second order linear

in fact we cannot get cycling solutions w/o second order eqns.

If the spring were forced: $mx'' + cx' + kx = f(x)$

Difference Between forced and unforced systems $\longrightarrow$

HOMOGENEOUS VS NON HOMOGENEOUS

if $F(x) = 0$ homogeneous
$F(x) \neq 0$ non homogeneous.

Classify the following systems as homogeneous or nonhomogeneous. and in general.

1. $y'' + x^3 y + 3y = 0$

2. $yy'' + y - 2y^2 = 0$

3. $y'' + \sqrt{x}\, y' + y = x^2 + 3$

4. $y'' + 3y' + y - 4 = 0$

The Principle of Superposition for Linear Systems.
- start w/ homogeneous case.

Thm Let y_1 and y_2 be solutions to a linear homogeneous equation on the interval I. If c_1 and c_2 are constants then the linear combination

$y = c_1 y_1 + c_2 y_2$ is also a solution.

Consider the general form:

$$y'' + p(x) y' + q(x) y = 0$$

assume y_1 and y_2 are solutions ... you try to prove Thm.

If y_1 is a soln then $y_1'' + p(x) y_1' + q(x) y = 0$
If y_2 is a soln then $y_2'' + p(x) y_2' + q(x) y_2 = 0$

consider $y = y_1 + y_2$ sub it into the eqn:

$$(y_1 + y_2)'' + p(x)(y_1 + y_2)' + q(x)(y_1 + y_2) = 0$$

$$y_1'' + y_2'' + p(x) y_1' + p(x) y_2' + q(x) y_1 + q(x) y_2 = 0$$

$$[y_1'' + p(x) y_1' + q(x) y_1] + [y_2'' + p(x) y_2' + q(x) y_2] = 0$$

$$0 + 0 = 0 \qquad \text{true!}$$

We write the <u>GENERAL SOLN</u> as the sum of all possible solns.

$$y = c_1 y_1 + c_2 y_2$$

To find a particular soln we need some initial conditions.

<u>Ex</u> consider the eqn: $y'' + y = 0$

It has the solutions $y_1 = \sin x \quad y_2 = \cos x$

- Write down the general soln.

$$y = A\sin x + B\cos x$$

Now assume $y(a) = b_0$
$y'(a) = b_1$

$$y(a) = A\sin(a) + B\cos(a) = b_0$$
$$y'(a) = A\cos(a) - B\sin(a) = b_1$$

system of linear eqns for A, B.

$$\begin{bmatrix} \sin(a) & \cos(a) \\ \cos(a) & -\sin(a) \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} b_0 \\ b_1 \end{bmatrix}$$

– the inverse must exist for A and B to have a solution!

This means that $\det \begin{vmatrix} \sin(a) & \cos(a) \\ \cos(a) & -\sin(a) \end{vmatrix} \neq 0$

OR that our two solutions must be linearly Independent!

<u>LINEARLY INDEPENDENT</u> two solutions are linearly independent provided that neither is a constant multiple of the other

* f/g and g/f not equal to a constant

$\sin x$ and $\cos x$ are linearly independent!

So, What do we have:

Given $y'' + p(x)y' + g(x)y = 0$ if we can find two linearly independent solutions y_1 and y_2 then we can write our general solution as

$$y = c_1y_1 + c_2y_2$$

and apply our initial conditions $y(a) = b_0$ and $y'(a) = b_1$ to find our particular solution

ASIDE The Wronskian Test for Linear Independence:

$$\text{Wronskian} = W = \begin{vmatrix} f & g \\ f' & g' \end{vmatrix} = fg' - f'g$$

if $W(f,g) = 0$ f and g are linearly dependent
if $W(f,g) \neq 0$ f and g are linearly independent.

Test the following functions using $W(y_1, y_2)$

1. $y_1 = \sin x \qquad y_2 = \cos x$
2. $y_1 = x \qquad y_2 = 2x$
3. $y_1 = e^x \qquad y_2 = xe^x$

Next Time → Prove that given a second order eqn
If we have found two solutions
we have found all solns.