Lecture 10 Differential Eqns.

- HW 8 Due Friday
- Test Corrections Due Friday
- Extended office hrs. TTh: 11-2.
- HW 9 Due Monday.

Last Time: Considered general higher order eqns:

$$y'' + P(x)y' + q(x)y = f(x) \qquad (\star)$$

$f(x) = 0$ homogeneous $\quad f(x) \neq 0$ non homogeneous.

<u>Principle of Superposition</u>: If y_1 and y_2 are solns then $y = c_1y_1 + c_2y_2$ are solns.

<u>GENERAL SOLN</u> $\quad y = c_1y_1 + c_2y_2$

For a particular soln to exist: y_1 and y_2 must be <u>linearly independent</u>.

How do we know that if we found two solutions to a second order linear eqn that we found <u>all</u> solns?

<u>Proof</u> <u>Assume</u> that $F = c_1y_1 + c_2y_2$ is a soln to $(\star)$ where y_1 and y_2 are linearly independent.
<u>Assume</u> that $G = c_3y_3 + c_4y_4$ is also a solution to $(\star)$ where y_3 and y_4 are L.I.

Let us try to find particular solutions such that $y(a) = b_0$ and $y'(a) = b_1$

$$\begin{aligned} c_1y_1 + c_2y_2 &= b_0 \\ c_1y_1' + c_2y_2' &= b_1 \end{aligned} \longrightarrow \begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} b_0 \\ b_1 \end{bmatrix}$$

$$\begin{aligned} c_3y_3 + c_4y_4 &= b_0 \\ c_3y_3' + c_4y_4' &= b_1 \end{aligned} \longrightarrow \begin{bmatrix} y_3 & y_4 \\ y_3' & y_4' \end{bmatrix}\begin{bmatrix} c_3 \\ c_4 \end{bmatrix} = \begin{bmatrix} b_0 \\ b_1 \end{bmatrix}$$

so $\begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} b_0 \\ b_1 \end{bmatrix} = \begin{bmatrix} y_3 & y_4 \\ y_3' & y_4' \end{bmatrix}\begin{bmatrix} c_3 \\ c_4 \end{bmatrix}$

$$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix}^{-1}\begin{bmatrix} b_0 \\ b_1 \end{bmatrix} = \begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix}^{-1}\begin{bmatrix} y_3 & y_4 \\ y_3' & y_4' \end{bmatrix}\begin{bmatrix} c_3 \\ c_4 \end{bmatrix}$$

only true if $\begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix}^{-1}\begin{bmatrix} y_3 & y_4 \\ y_3' & y_4' \end{bmatrix} = I$

thus $y_1 = y_3$ and $y_2 = y_4$.

All of this can be generalized to higher order linear systems!

$$P_0(x)y^{(n)} + P_1(x)y^{(n-1)} + \cdots + P_{n-1}(x)y' + P_n(x)y = F(x)$$

we need to find n- linearly independent solutions and use the principle of soperposition to write the general soln

$$y(x) = c_1y_1 + c_2y_2 + \cdots + c_ny_n$$

<u>And</u> we can use the Wronskian to test for: Linear Independence

$$W = \begin{vmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & \cdots & y_n' \\ \vdots & & & \vdots \\ y_1^{(n-1)} & y_2^{n-1} & \cdots & y_n^{n-1} \end{vmatrix}$$

As long as $P_n(x)$'s are continuous "nice" functions a solution exists and is unigre!

LINEAR HOMOGENEOUS EQUATIONS w/ CONSTANT COEF.

$$ay'' + by' + cy = 0$$ spring-mass problem.

GOAL – find a general solution method.

Think about functions... what function do we take the derivative of and just get a constant multiple back?

$$y = e^{rx} \text{ then } y' = re^{rx} \text{ and } y'' = r^2 e^{rx}$$

this kinda replicates $ay'' + by' + cy$ ...

Lets look for solutions of the form $y = e^{rx}$
plug this in to ODE and see what happens.

$$ar^2 e^{rx} + bre^{rx} + ce^{rx} = 0$$

so

$$e^{rx}\left[ar^2 + br + c\right] = 0 \qquad e^{rx} \neq 0$$

so $ar^2 + br + c = 0$ if we can solve for r we have a soln.

This is called the Characteristic Equation (auxilliary eqn)

We have reduced the problem of solving a second order equation down to solving an algebraic eqn!

In the most general case:

$$\text{CE:} \quad a_n r^n + a_{n-1} r^{n-1} + \cdots + a_1 r + a_0 = 0$$

Fundamental Thm of Algebra $\rightarrow$ Every n^{th} degree polynomial has n-zeros. The zeros are not necessarily real or distinct but they exist!

We have three possible cases: $ar^2 + br + c = 0$

1. Two distinct REAL roots $r = r_1$ $r = r_2$
   $(x-1)(x-2) = 0$
2. One repeated root
   $(x-1)^2 = 0$ $r = r_1$
3. Complex roots
   $r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ $b^2 - 4ac < 0$

CONSIDER EACH CASE:

$\boxed{\text{DISTINCT REAL ROOTS}}$ If we find distinct real roots then the solution has the form:

$$y = \underbrace{c_1 e^{r_1 x} + c_2 e^{r_2 x}}_{\text{second order}} + \cdots + c_n e^{r_n x}$$

second order
we have just two roots.

Ex Solve the Initial Value Problem: $y^{(3)} + 3y'' - 10y' = 0$
$y(0) = 7$ $y'(0) = 0$ $y''(0) = 70$

second order, linear constant coef.

how many conditions?

1. Write down the characteristic equation:

$$r^3 + 3r^2 - 10r = 0$$

→ I can just read this from the ODE

2. Solve for all possible roots.

$$r(r^2 + 3r - 10) = r(r+5)(r-2) = 0$$

$r = 0$ $r = -5$ $r = 2$
three distinct real roots

3. Write down general soln

$$y = c_1 e^{2x} + c_2 e^{-5x} + c_3$$

4. Use initial conditions to find c_1 c_2 and c_3:

$$y(0) = c_1 + c_2 + c_3 = 7$$
$$y'(0) = 2c_1 - 5c_2 = 0$$
$$y''(0) = 4c_1 + 25c_2 = 70$$

Three eqns
three unknowns.

Ways to solve these:

Substitution and row reduction $\longrightarrow$

$$\begin{aligned} C_1 + C_2 + C_3 &= 7 \quad &A.\\ 2C_1 - 5C_2 &= 0 \quad &B.\\ 4C_1 + 25C_2 &= 70 \quad &C. \end{aligned}$$

Take $5B + C$

$$\begin{aligned} 10C_1 - 25C_2 &= 0\\ + \; 4C_1 + 25C_2 &= 70\\ 14C_1 &= 70 \end{aligned} \qquad C_1 = \frac{70}{14} = 5 \qquad \boxed{C_1 = 5}$$

sub into B $\quad 2C_1 - 5C_2 = 10 - 5C_2 = 0 \quad \boxed{C_2 = 2}$

sub into A $\quad C_1 + C_2 + C_3 = 5 + 2 + C_3 = 7 + C_3 = 7$

$$\boxed{C_3 = 0}$$

Particular Soln: $\quad y(x) = 5e^{2x} + 2e^{-5x}$

Matrix Inverse $\longrightarrow$

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & -5 & 0 \\ 4 & 25 & 0 \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \\ C_3 \end{bmatrix} = \begin{bmatrix} 7 \\ 0 \\ 70 \end{bmatrix}$$

find the Inverse.

ON MATLAB ...

CALCULATOR ...

OR BY HAND:

$$A^{-1} = \frac{1}{\det(A)}\left[\operatorname{adj}(A)\right]$$

this is labor intensive but works.

$$\operatorname{adj}(A) = \begin{bmatrix} M_{11} & -M_{12} & M_{13} \\ -M_{21} & M_{22} & -M_{23} \\ M_{33} & -M_{32} & M_{33} \end{bmatrix}^T$$

$M^T \Rightarrow$ transpose.

where M_{ij} is the det of Cofactor matrix

$$M_{11} = \begin{vmatrix} -5 & 0 \\ 25 & 0 \end{vmatrix} \qquad M_{12} = \begin{vmatrix} 2 & 0 \\ 4 & 0 \end{vmatrix}$$

and so on ...

Review: Matrix multiplication:

(rows) x (col) $\underset{n \times m}{\begin{bmatrix} \longrightarrow \\ \longrightarrow \\ \longrightarrow \end{bmatrix}} \underset{m \times k}{\begin{bmatrix} \downarrow & \downarrow & \downarrow \end{bmatrix}}$

$m = m$
new matrix
is $n \times k$

# of rows in second matrix
must match # of collumns in first!

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} ae+bg & af+bh \\ ce+dg & cf+dh \end{bmatrix}$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} e \\ g \end{bmatrix} = \begin{bmatrix} ae+bg \\ ce+dg \end{bmatrix}$$

ON MATLAB $\rightarrow$ SOLVE FOR c_1 c_2 c_3 ...

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A(1,:) = [1,1,1]
A(2,:) = [2,-5,0]
A(3,:) = [4,25,0]

b = [7,0,70]
b = transpose(b)

C = inv(A)*b.
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You Try:

$y'' - 4y = 0$
$y(0) = 1 \quad y'(0) = 0$

$2y'' - 3y' = 0$
$y(0) = 0 \quad y'(0) = 0$