- Hand in HW 8 / Test Corrections
- HW9 due Mon
- HW 10 due Wed.

Last Time: Linear Higher Order Constant Coef.

$$a_n y^{n} + a_{n-1} y^{n-1} + \cdots a_2 y' + a_1 y = 0 \quad \text{homogeneous.}$$

Solution Method: 1. Write down the Characteristic Equation
- read directly from ODE
- OR sub in $y = e^{rx}$

$$a_n r^n + a_{n-1} r^{n-1} + \cdots + a_2 r + a_1 = 0$$

2. Find and Classify the roots r_i's of the Characteristic Eqn.
3. Construct Solution Based on roots and $y = e^{rx}$ [Sum of Solns]
4. Apply I.C.'s if given.

EX $y^{(4)} + 4y^{(3)} = 0$

how many linearly independent solutions do I expect? FOUR.
unique soln? YES.

CHARACTERISTIC EQN $r^4 + 4r^3 = 0$

$r^3(r+4) = 0 \qquad r = 0$ and $r = -4$

does this only have two roots?
No it must have 4
one root @ $r = -4$
three roots @ $r = 0$

So can we write $y(x) = c_1 e^{-4x} + c_2 e^0 + c_3 e^0 + c_4 e^0$

$= c_1 e^{-4x} + A$

only two solns... this is a problem.

CASE 2 repeated roots... consider a simpler example to start:

EX $y'' + 2y' + y = 0$

1. Characteristic eqn: $r^2 + 2r + 1 = 0$

$$(r+1)(r+1) = 0$$

2. (solve) $r = -1$ is a double root.

3. Write down eqn for general soln.

$$y(x) = c_1 e^{-x} \rightsquigarrow \text{problem: we need another solution.}$$

THE FIX: assume $y = u(x)e^{-x}$ → I look for solutions like this... plug in and see what happens.

$$y' = u'(x)e^{-x} - u(x)e^{-x}$$
$$y'' = u''(x)e^{-x} - u'(x)e^{-x} - u'(x)e^{-x} + u(x)e^{-x}$$
$$= u''(x)e^{-x} - 2u'(x)e^{-x} + u(x)e^{-x}$$

Plug in: $u''(x)e^{-x} - \cancel{2u'(x)e^{-x}} + \cancel{u(x)e^{-x}} + \cancel{2u'(x)e^{-x}} - \cancel{2u(x)e^{-x}} + \cancel{u(x)e^{-x}} =$

$$u''(x)e^{-x} = 0 \qquad u''(x) = 0 \quad \text{integrate twice}$$

$$u(x) = Ax + b$$

$$y(x) = (Ax+b)e^{-x} = c_1 e^{-x} + c_2 x e^{-x}$$

↑ second linearly independent term found by mult by x

IN GENERAL.

Given a root r_1 to the characteristic equation with multiplicity k the part of the soln corresponding to r_1 is written

$$(c_1 + c_2 x + c_3 x^2 + \cdots + c_k x^{k-1})e^{r_1 x}$$

Back to our initial example: $y^{(4)} + 4y^{(3)} = 0$

$r = -4$

$r = 0$ triple root

$$y(x) = c_1 e^{-4x} + (c_2 + c_3 x + c_4 x^2) e^{0x}$$
$$= c_1 e^{-4x} + c_2 + c_3 x + c_4 x^2$$

Construct a solution given the following Characteristic Equations:

A. $(r+1)^4 (r-1)(r-2) = 0$

$$y(x) = c_1 e^{x} + c_2 e^{2x} + (c_3 + c_4 x + c_5 x^2 + c_6 x^3) e^{-x}$$

hint: count terms. one for each root!

B. $(r-2)^4 (r-1)^3 (r+1) = 0$

$$y(x) = c_1 e^{-x} + (c_2 + c_3 x + c_4 x^2) e^{x} + (c_5 + c_6 x + c_7 x^2 + c_8 x^3) e$$

C. Find the general solution to $9y^{(5)} - 6y^{(4)} + y^{(3)} = 0$

$r^3 (3r-1)(3r-1) = 0$ $\quad y(x) = (c_1 + c_2 x) e^{1/3 x} + (c_3 + c_4 x + c_5 x^2)$

<u>CASE 3</u> Complex roots.

Imagine we want to solve $y'' + y = 0$

CE $r^2 + 1 = 0$ $\quad r = \pm\sqrt{-1} = \pm i$

so our solution: $y(x) = c_1 e^{-ix} + c_2 e^{ix}$

there is a better way to ~~do~~ write this answer.

EULERS FORMULA $\quad e^{ix} = \cos x + i \sin x$

Where does this come from?

Taylor Series for e^t

$$e^t = \sum_{n=0}^{\infty} \frac{t^n}{n!} = 1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \frac{t^4}{4!} + \cdots$$

in e^{ix} then $t = ix$

$$e^{ix} = 1 + (ix) + \frac{(ix)^2}{2!} + \frac{(ix)^3}{3!} + \frac{(ix)^4}{4!} + \cdots$$

but $i^2 = -1$

$$= 1 + ix - \frac{x^2}{2!} - \frac{ix^3}{3!} + \frac{x^4}{4!} + \cdots$$

gather terms.

$$= \underbrace{\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots\right)}_{\text{taylor series } \cos x} + i\underbrace{\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots\right)}_{\text{taylor series } \sin x}$$

so $e^{ix} = \cos x + i\sin x$

our solution was $y(x) = c_1 e^{ix} + c_2 e^{-ix}$

$$= c_1(\cos x + i\sin x) + c_2(\cos x - i\sin x)$$

$$= (c_1 + c_2)\cos x + i(c_1 - c_2)\sin x = A\cos x + B\sin x$$

General Complex Form: $r = a \pm bi$ complex conjugate pair.

real part imaginary part

$$y(x) = c_1 e^{a+bi} + c_2 e^{a-bi} = e^a\left(c_1 e^{bi} + c_2 e^{-bi}\right)$$

$$= e^a\left(A\sin(bx) + B\cos(bx)\right)$$

ASIDE: When we have imaginary solns $r = bi$ we sometimes write our soln as

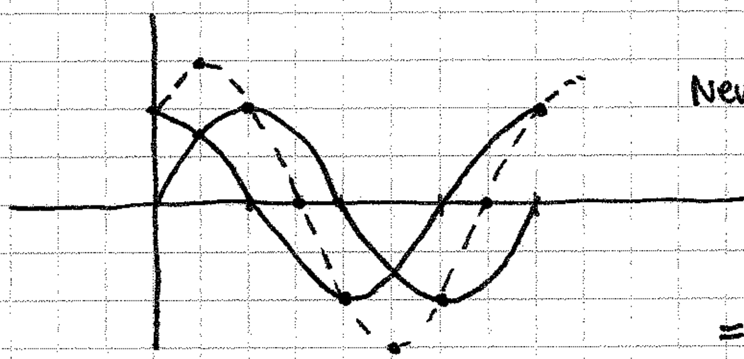
$$y(x) = A\sin(bx + \varphi)$$

A = amplitude
φ = phase shift.

<u>Why:</u> Given $y(x) = A\cos(bx) + B\sin(bx)$

we can write $y(x) = A\sin(bx + \frac{\pi}{2}) + B\sin(bx)$

$\cos x$ is just $\frac{\pi}{2}$ shift away from $\sin x$



New wave
same period
different amplitude
and phase shift

$= A\sin(bx + \varphi)$

initial conditions give
A and φ.

<u>You Try:</u> $y'' - 4y' + 5y = 0$
$y(0) = 1 \quad y'(0) = 5$

$$y(x) = e^{2x}(\cos x + 3\sin x)$$

What if the roots are $0, 0, 3 \pm 2i, 3 \pm 2i$
repeated complex

HINT APPLY SAME PATTERN!

$$y(x) = C_1 x + C_2 + e^{2x}(C_1 + C_2 x)(C_3\cos 3x + C_4\sin 3x)$$

<u>Ex</u> Find the general Soln: $y^{(3)} + y' - 10y = 0$

$$r^3 + r - 10 = 0$$

How do we solve for roots of a cubic.

1. only possible roots are factors of constant term
$\pm 1, \pm 2, \pm 5$ and ± 10.

Guess and check $\quad r=1 \quad 1+1-10 \neq 0$

$\qquad\qquad\qquad\qquad r=-1 \quad -1-1-10 \neq 0$

$\qquad\qquad\qquad\qquad r=2 \quad 8+2-10 = 0$ ✓

$r=2$ is a root ... factor out $(r-2)$

$$
\begin{array}{r|l}
 & r^2+2r+5 \\
\hline
r-2 & r^3+0r^2+r-10 \\
 & -(r^3-2r^2) \\
\hline
 & \quad 2r^2+r-10 \\
 & \quad -(2r^2-4r) \\
\hline
 & \qquad 5r-10 \\
 & \qquad 5r-10 \\
\hline
 & \qquad\quad 0
\end{array}
$$

you should have zero remainder when factoring out roots.

$$(r-2)(r^2+2r+5) = (r-2)(\ ?\ )(\ ?\ )$$

us quadratic formula

$$r = \frac{-2 \pm \sqrt{4-4\cdot 5}}{2} = -1 \pm \sqrt{1-5} = -1 \pm 2i$$

complex root.

$$y(x) = c_1 e^{2x} + e^{-x}\left(c_2 \sin(2x) + c_3 \cos(2x)\right)$$

<u>More to try:</u>

$$y^{(3)} + 8y'' + 21y + 18 = 0 \qquad y(x) = c_1 e^{-2x} + (c_2 + c_3 x)e^{-3x}$$

$$y'' - 6y + 13 = 0 \qquad r = 3 \pm 2i \qquad y(x) = e^{3x}(c_1 \sin 2x + c_2 \cos 2x)$$