Lecture 13 Differential Eqns.

- Hand in HW 9
- HW 10 due Wed
- HW 11 due Friday.

- Take 5 min → look back over notes
  rewrite rules for higher order
  constant coef. eqns.

1. Real Distinct Roots: r_1, r_2 $\quad y(x) = c_1 e^{r_1 x} + c_2 e^{r_2 x}$
2. Real Repeated Roots: r_1, r_1 $\quad y(x) = c_1 e^{r_1 x} + c_2 x e^{r_1 x}$
3. Complex Roots: $a \pm bi$ $\quad y(x) = e^{ax}(A \sin bx + B \cos(bx))$

   or

   $y(x) = A e^{ax} \sin(bx + \varphi)$.

3.5 Non-homogeneous Equations w/ Constant Coef.

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 y' + a_0 y = f(x)$$

OPERATOR NOTATION $\Rightarrow$ $L = a_n \dfrac{d^n}{dx^n} + a_{n-1} \dfrac{d^{n-1}}{dx^{n-1}} + \cdots + a_1 \dfrac{d}{dx} + a_0$

so we write our linear system: $Ly = f(x)$.

Homogeneous Case $\quad Ly_H = 0 \quad$ Find roots

write soln $y_H = c_1 y_1 + c_2 y_2 + \cdots + c_n y_n$

NOW CONSIDER $\quad Ly = f(x)$

GOAL: Write down a general solution to this equation.

We separate out our solution into a homogeneous and nonhomogeneous part

$$y = y_H + y_{NH}$$

where $Ly_H = 0$ and $Ly_{NH} = f(x)$

$$Ly = L(y_H + y_{NH}) = Ly_H + Ly_{NH} = 0 + f(x) = f(x)$$

most general soln will include homogeneous terms!

Sometimes: Homogeneous Soln $\longrightarrow$ Complimentary Soln
NonHomogeneous Soln $\longrightarrow$ Particular Soln.

We already know how to find y_H ... how do we find y_{NH}.

THE METHOD OF UNDETERMINED COEFICIENTS.

- we make an educated guess for y_{NH} with flexibility in the coef.
- then solve for the coef.

If $f(x)$ is a polynomial of degree m $\rightarrow$ we guess a general polynomial of degree m.

If $f(x)$ is $\sin(x)$ or $\cos(x)$ we guess $\sin(x) + \cos(x)$

<u>EX</u> $y'' + 3y' + 4y = 3x + 2.$

1. Find the homogeneous part: $y_H'' + 3y_H' + 4y_H = 0$

CE: $r^2 + 3r + 4 = 0$

$$r = \frac{-3 \pm \sqrt{9-16}}{2} = -3/2 \pm \frac{\sqrt{7}i}{2}$$

$$y_H = e^{-3x/2}\left[A\sin\left(\frac{\sqrt{7}x}{2}\right) + B\cos\left(\frac{\sqrt{7}x}{2}\right)\right]$$

2. Find the nonhomogeneous solution

A. Look at $f(x)$: $f(x) = 3x + 2$

B. Guess y_{NH} is similar to $f(x)$

$$y_{NH} = Ax + B.$$

c. Sub your guess into the ODE and solve for the coeficients.

$$y_{NH}' = A \qquad y_{NH}'' = 0$$

$$0 + 3A + 4Ax + 4B = 3x + 2$$
$$4Ax + (3A + 4B) = 3x + 2$$

$$4A = 3 \qquad 3(3/4) + 4B = 2$$
$$A = 3/4 \qquad 4B = 2 - 9/4$$
$$B = -1/16$$

so $y_{NH} = 3/4\, x - 1/16.$

3. Put y_H and y_{NH} back together to form the general solution.

$$y = y_H + y_{NH} = e^{-3/2x}\left[A\sin\left(\frac{\sqrt{7}x}{2}\right) + B\cos\left(\frac{\sqrt{7}x}{2}\right)\right] + \tfrac{3}{4}x - \tfrac{1}{6}.$$

<u>EX</u> $y'' - 4y = 2e^{3x}$

1 Homogeneous $y_H'' - 4y_H = 0 \qquad r^2 - 4 = 0 \qquad r = \pm 2$

$$y_H = c_1 e^{2x} + c_2 e^{-2x}$$

2. Nonhomogeneous: $y_{NH}'' - 4y_{NH} = f(x)$

A. $f(x) = 2e^{3x}$ B. guess $y_{NH} = Ae^{3x}$

c. $y'_{NH} = 3Ae^{3x}$ $\quad y''_{NH} = 9Ae^{3x}$

$$9Ae^{3x} - 4Ae^{3x} = 2e^{3x}$$

$$5Ae^{3x} = 2e^{3x} \qquad 5A = 2 \qquad A = 2/5$$

$$y_{NH} = 2/5\, e^{3x}$$

3. Put the general soln together:

$$y(x) = Ae^{2x} + Be^{-2x} + 2/5\, e^{3x}.$$

NOTE $\rightarrow$ If IC's are given ONLY APPLY AFTER you find the general soln!

<u>TRY:</u> $y'' - y = \sin(2x)$

$y_H = Ae^{x} + Be^{-x}$

Hint:
guess $y_{NH} = A\sin(2x) + B\cos(2x)$

$y''_{NH} = -4A\sin(2x) - 4B\cos(2x)$

$$-4A\sin(2x) - 4B\cos(2x) - A\sin(2x) - B\cos(2x) = \sin(2x)$$

$$-3A\sin(2x) - 3B\cos(2x) = \sin(2x)$$

$-3A = 1 \qquad -3B = 0$

$A = -1/3 \qquad B = 0$

$$y(x) = Ae^{-x} + Be^{x} - 1/3\sin(2x)$$

<u>Ex</u> What would you guess if $y_H = c_1e^{2x} + c_2e^{-2x}$ and ...

$f(x) = x^2$

$f(x) = e^{x} + \sin(x) + x$

$f(x) = 4x^3 + x$

TROUBLE MAKERS → The Case of Duplication.

Consider $y'' - 4y = 2e^{2x}$

1 $y_H'' - 4y_H = 0$ $\quad r^2 - 4 = 0 \quad r = \pm 2$

$$y_H = c_1 e^{2x} + c_2 e^{-2x}$$

2. Look at $f(x) = 2e^{2x}$

guess $y_{NH} = Ae^{2x}$ — But wait!! This duplicates the homogeneous.

If we plug this in we will get zero! $0 \neq 2e^{2x}$

A Better Guess $y_{NH} = Axe^{2x}$ → need another LI soln... similar to repeated roots

$$y_{NH}' = Ae^{2x} + 2Axe^{2x}$$
$$y_{NH}'' = 2Ae^{2x} + 2Ae^{2x} + 4Axe^{2x}$$
$$= 4Ae^{2x} + 4Axe^{2x}$$

$$4Ae^{2x} + 4Axe^{2x} - 4Axe^{2x} = 4Ae^{2x} = f(x) = 2e^{2x}$$

$$4A = 2 \qquad A = 1/2$$

3. Put it all back together

$$y(x) = c_1 e^{2x} + c_2 e^{-2x} + \tfrac{1}{2} x e^{2x}$$

Method of Undetermined Coeficients: $Ly = f(x)$

1. Find the homogeneous solution $Ly_H = 0$

2. Find the nonhomogeneous solution $Ly_{NH} = f(x)$

   A. Guess y_{NH} based off $f(x)$... It should be of similar form.

   $f(x) =$ polynomial degree m $\longrightarrow$ $y_{NH} = C_m x^m + C_{m-1} x^{m-1} + \cdots + Cx + C$

   $f(x) = \sin(ax)$ or $\cos(ax)$ $\quad y_{NH} = A\sin(ax) + B\cos(ax)$

   $f(x) = e^{bx}$ $\quad y_{NH} = Ae^{bx}$

   B. Compare your guess with y_H... if there is a duplication, make a better guess.

   multiply through by x^s $\rightarrow$ what ever power of x you need so there are NO more duplications

   C. Sub in your guess Ly_{NH} and set equal to $f(x)$... solve for the coeficients.

3. Write Full General Soln as $y = y_H + y_{NH}$

4. Apply Initial Conditions if Given.

Ex $y'' + 6y' + 13y = e^{-3x}\cos 2x + x$ Find the form of y_{NH}.

$$y_H = e^{-3x}(A\cos 2x + B\sin 2x)$$

<u>Guess</u> $$y_{NH} = \underbrace{e^{-3x}(A\cos 2x + B\sin 2x)}_{\text{Duplicates the homogeneous.}} + Dx + E$$

<u>Better Guess</u> $$y_{NH} = xe^{-3x}(A\cos 2x + B\sin 2x) + Dx + E$$

Ex $Ly = x^2 e^{2x} + x\sin 3x$ y_H has C.E $(r-2)^3(r^2+9)=0$

$$y_H = (C_1 + C_2 x + C_3 x^2)e^{2x} + C_4\cos 3x + C_5 \sin 3x$$

<u>Guess</u> $$y_{NH} = \underbrace{(C_1x^2 + C_2x + C_3)e^{2x}}_{\substack{\text{duplication}\\ \text{mult by } x^3}} + \underbrace{(C_4x + C_5)\sin 3x + (C_6 + C_7x)\cos 3x}_{\substack{\text{duplication.}\\ \text{mult by } x}}$$

<u>Better Guess</u> $$y_{NH} = x^3(C_1x^2 + C_2x + C_3)e^{2x} + x\left[(C_4x + C_5)\sin 3x + (C_6 + C_7x)\cos 3x\right]$$

NEXT TIME LOTS OF EXAMPLES!