Farlow Lesson 9.
Eigenfunction Expansions.

GOAL... solve equations of the form:

$$\begin{cases} u_t = \alpha^2 u_{xx} + f(x,t) \\ \alpha_1 u_x(0,t) + \beta_1 u(0,t) = 0 \\ \alpha_2 u_x(L,t) + \beta_2 u(L,t) = 0 \\ u(x,0) = \phi(x) \end{cases}$$

NON-HOMOGENEOUS PDE

HOMOGENEOUS BC's.

THE IDEA... if $f(x,t)=0$ then we have our typical diffusion equation. To solve we would separate variables.

$$u(x,t) = T(t)X(x)$$

and our soln would be:

$$u(x,t) = \sum_{n=1}^{\infty} A_n e^{-(\lambda_n \alpha)^2 t} X_n(x)$$

where we find λ_n and X_n by solving

$$X'' + \lambda^2 X = 0$$

subject to the boundary conditions.

OUR ASSUMPTION $\rightarrow$ The non-homogeneous term $f(x,t)$ is a source term ... that means that $T(t) = e^{-(\lambda_n\alpha)^2 t}$ can't just damp out to a steady state. but we assume our soln has the same basic form.

$$u(x,t) = \sum_{n=1}^{\infty} T_n(t) X_n(x)$$

plug into pde for these.

solve the associated eigenfunction problem for these

EX Do eigenfunction expansion in general for Direchlet Boundary Conditions.

NOTE

DIRECHLET BC's $\Rightarrow$ $u(0,t) = \alpha_1 \quad u(L,t) = \alpha_2$

NEUMANN BC's $\Rightarrow$ $u_x(0,t) = \beta_1 \quad u_x(L,t) = \beta_2$

ROBIN BC's $\Rightarrow$ $a_1 u_x(0,t) + b_1 u(0,t) = g_1 \quad a_2 u_x(L,t) + b_2 u(L,t) = g_2$

MIXED BC's $\Rightarrow$ combination of the above.

consider the equation:

$$(1)\quad \begin{cases} u_t = u_{xx} + f(x,t) & \text{Non-homogeneous PDE.} \\ u(0,t) = 0 \\ u(1,t) = 0 \\ u(x,0) = \phi(x) \end{cases}$$

Assume solution of the form $u(x,t) = \sum_{n=1}^{\infty} T_n(t) X_n(x)$
where $X_n(x)$ are the associated eigenfunctions.

The associated eigenfunction problem

$$X'' + \lambda^2 X = 0$$
$$X(0) = 0$$
$$X(1) = 0$$

where did this come from?

If we look @ the homogeneous problem + BC's

$$u_t = u_{xx}$$
$$u(0,t) = 0 \quad \Rightarrow \quad \text{then separate variables}$$
$$u(1,t) = 0 \qquad\qquad u = X(x)T(t)$$

PDE:

$$XT' = X''T$$

so $\dfrac{T'}{T} = \dfrac{X''}{X} = k$

$$[T' - kT = 0] + [X'' - kX = 0]$$

BC: $T(t)X(0) = 0 \;\Rightarrow\; X(0) = 0$
$T(t)X(1) = 0 \qquad X(1) = 0$

$k > 0$ then $X(x) = Ae^{-\sqrt{k}x} + Be^{\sqrt{k}x}$

$$X(0) = A + B = 0 \quad \rightarrow \quad A = B = 0$$
$$X(1) = Ae^{-\sqrt{k}} + Be^{-\sqrt{k}} = 0$$

ACTUALLY NEVER GET POSITIVE K VALUES SO WE LET $k = -\lambda^2$

And our e-val/function problem becomes

$$X'' + \lambda^2 X = 0$$
$$X(0) = 0 \quad \Rightarrow \quad X(x) = A\sin(\lambda x) + B\cos(\lambda x)$$
$$X(1) = 0$$

$$X_n = \sin(n\pi x) \qquad \lambda_n = n\pi$$

so we assume $u(x,t) = \sum_{n=1}^{\infty} T_n(t)\sin(n\pi x)$

sub this into our PDE (1).

PDE
$$\begin{cases} u_t = \sum_{n=1}^{\infty} T_n' \sin(n\pi x) \\ u_{xx} = \sum_{n=1}^{\infty} -T_n \sin(n\pi x)(n\pi)^2 \end{cases}$$

BC's
$$\begin{cases} u(0,t) = \sum_{n=1}^{\infty} T_n(t)\sin(0) = 0 = 0 \checkmark \\ u(1,t) = \sum_{n=1}^{\infty} T_n(t)\sin(n\pi) = 0 = 0 \checkmark \end{cases}$$

the BC's should always be automatically satisfied by $X_n(x)$.

IC's

$$u(x,0) = \sum_{n=1}^{\infty} T_n(0)\sin(n\pi x) = \phi(x)$$

use orthogonality to solve for the $T_n(0)$'s.

$$\int_0^1 \sum_{n=1}^{\infty} T_n(0)\sin(n\pi x)\sin(m\pi x)\,dx = \int_0^1 \phi(x)\sin(m\pi x)\,dx$$

orthogonality $m=n$

$$\frac{T_n(0)}{2} = \int_0^1 \phi(x)\sin(m\pi x)\,dx$$

$$T_n(0) = 2\int_0^1 \phi(x)\sin(m\pi x)\,dx$$

NEW PDE

$$\sum_{n=1}^{\infty} T_n' \sin(n\pi x) = -\alpha^2 \sum_{n=1}^{\infty} (n\pi)^2 T_n \sin(n\pi x) + f(x,t)$$

again mult by $\sin(m\pi x)$ and use orthogonality to get T_n terms outside of sums.

$$\int_0^1 \sum_{n=1}^{\infty} T_n' \sin(n\pi x)\sin(m\pi x)\,dx = -\alpha^2 \int_0^1 \sum_{n=1}^{\infty} (n\pi)^2 T_n \sin(n\pi x)\sin(m\pi x)\,dx$$

$m=n$ and $\int_0^1 \sin^2(m\pi x)\,dx = 1/2$ $\qquad + \int_0^1 f(x,t)\sin(m\pi x)\,dx$

$$\frac{T_n'}{2} = -\alpha^2 (n\pi)^2 \frac{T_n}{2} + \int_0^1 f(x,t)\sin(m\pi x)\,dx$$

$$T_n' + (\alpha n\pi)^2 T_n = \underbrace{2\int_0^1 f(x,t)\sin(m\pi x)\,dx}_{}$$

this just gives some function of t ... call it $g_n(t)$

ODE for the T_n's.
$$\begin{cases} T_n' + (\alpha n\pi)^2 T_n = g_n(t). \\ T_n(0) = 2\int_0^1 \phi(x)\sin(m\pi x)\,dx \end{cases}$$

First order linear ODE ... we can use method of integrating factor.
$$\frac{dy}{dx} + P(x)y = Q(x)$$
similar form

$$\rho(t) = e^{\int P(t)dt} = e^{\int(\alpha n\pi)^2 dt} = e^{(\alpha n\pi)^2 t}$$

$$\left[e^{(\alpha n\pi)^2 t} T_n\right]' = e^{(\alpha n\pi)^2 t} g_n(t)$$
mult by integrating factor and write LHS as derivative of a product.

$$e^{(\alpha n\pi)^2 t} T_n = \int_0^t e^{(\alpha n\pi)^2 \tau} g_n(\tau)\,d\tau + A_n$$
$A_n \longrightarrow$ constant of integration

$$T_n = e^{-(\alpha n\pi)^2 t}\int_0^t e^{(\alpha n\pi)^2\tau} g_n(\tau)\,d\tau + A_n e^{-(\alpha n\pi)^2 t}$$

Apply IC's to get A_n's.
write full solution as $u(x,t) = \sum_{n=1}^{\infty} T_n(t)\sin(n\pi x)$

A SPECIFIC EXAMPLE.

$$\begin{cases} u_t = \alpha^2 u_{xx} + \sin(3\pi x) \\ u(0,t) = 0 \\ u(1,t) = 0 \\ u(x,0) = \sin(\pi x) \end{cases}$$

assume $u = \sum_{n=1}^{\infty} T_n(t) X_n(x)$

associated eigenfunction problem
$$X'' + \lambda^2 X = 0$$
$$X(0) = 0 \quad X(1) = 0$$
$$X_n(x) = \sin(n\pi x)$$

ASSUME: $u(x,t) = \sum_{n=1}^{\infty} T_n(t)\sin(n\pi x)$

SUB INTO PDE ...
$$u_t = \sum_{n=1}^{\infty} T_n' \sin(n\pi x) \qquad u_{xx} = \sum_{n=1}^{\infty} -(n\pi)^2 T_n \sin(n\pi x)$$

so
$$\sum_{n=1}^{\infty} T_n' \sin(n\pi x) = -\alpha^2 \sum_{n=1}^{\infty} (n\pi)^2 T_n \sin(n\pi x) + \sin(3\pi x)$$

$$\int_0^1 \sum_{n=1}^{\infty} T_n' \sin(n\pi x)\sin(m\pi x)\,dx = -\alpha^2 \int_0^1 \sum_{n=1}^{\infty} (n\pi)^2 T_n \sin(n\pi x)\sin(m\pi x)\,dx$$
$$+ \int_0^1 \sin(3\pi x)\sin(m\pi x)\,dx$$

by orthogonality: $\frac{T_n'}{2} = -(\alpha n\pi)^2 \frac{T_n}{2} + \int_0^1 \sin(3\pi x)\sin(m\pi x)\,dx$

call $g_n = \int_0^1 \sin(3\pi x)\sin(n\pi x)\,dx = \begin{cases} 1/2 & n=3 \\ 0 & \text{otherwise} \end{cases}$

OUR NEW PROBLEM $\quad T_n' + (\alpha n\pi)^2 T_n = 2g_n \quad$ linear, first order, ODE.

SUB INTO IC's $\quad u(x,0) = \sum_{n=1}^{\infty} T_n(0)\sin(n\pi x) = \sin(\pi x)$

use orthogonality to get T_n's alone.

$$\int_0^1 \sum_{n=1}^{\infty} T_n(0)\sin(n\pi x)\sin(m\pi x)\,dx = \int_0^1 \sin(\pi x)\sin(m\pi x)\,dx$$

$$\frac{T_n(0)}{2} = \int_0^1 \sin(\pi x)\sin(m\pi x)\,dx = \begin{cases} 1/2 & m=1 \\ 0 & \text{otherwise} \end{cases}$$

$$T_n(0) = \begin{cases} 1 & m=1 \\ 0 & \text{otherwise.} \end{cases}$$

$$\begin{cases} T_n' + (\alpha n\pi)^2 T_n = 2g_n & g_n = \begin{cases} 1/2 & n=3 \\ 0 & \text{otherwise} \end{cases} \\ T_n(0) = \begin{cases} 1 & m=1 \\ 0 & \text{otherwise} \end{cases} \end{cases}$$

This is a infinate # of ODE's one for each $n=1,2,3\ldots$ look at them one at a time and see if there is a pattern.

$n=1 \quad T_1' + (\alpha\pi)^2 T_1 = 0 \quad$ seperable first order ODE

$T_1(0) = 1 \quad \Rightarrow \quad T_1(t) = Ae^{-(\alpha\pi)^2 t}$

$T_1(0) = A = 1$

$$T_1(t) = e^{-(\alpha\pi)^2 t}.$$

$n=2 \quad T_2' + (2\alpha\pi)^2 T_2 = 0 \quad \Rightarrow \quad$ Same ODE as $n=1$

$T_2(0) = 0 \qquad T_2(t) = Ae^{-(2\alpha\pi)^2 t}$

$T_2(0) = A = 0$

$$T_2(t) = 0$$

$n=3 \quad T_3' + (3\alpha\pi)^2 T_3 = 1$

$T_3(0) = 0$

linear first order

could use integrating factor

OR treat it as non homogeneous.

with $T_{3_H} = A e^{-(3\alpha\pi)^2 t}$

guess $T_{3p} = C =$ constant

$T_{3p}' = 0$

so $(3\alpha\pi)^2 T_3 = 1 \qquad C = 1/(3\alpha\pi)^2$

$$T_3 = Ae^{-(3\alpha\pi)^2 t} + 1/(3\alpha\pi)^2$$

$$T_3(0) = A + 1/(3\alpha\pi)^2 = 0 \qquad A = -1/(3\alpha\pi)^2$$

$$T_3(t) = \frac{1}{(3\alpha\pi)^2}\left[1 - e^{-(3\alpha\pi)^2 t}\right]$$

$n = 4,5,6\ldots \quad T_n' + (n\alpha\pi)^2 T_n = 0 \quad \Rightarrow \quad T_n(t) \equiv 0$

$T_n(0) = 0$

NOW WE RECONSTRUCT OUR SOLN

$$u(x,t) = \sum_{n=1}^{\infty} T_n(t)\sin(n\pi x)$$

$$= e^{-(\alpha\pi)^2 t}\sin(\pi x) + \frac{1}{(3\alpha\pi)^2}\left[1 - e^{-(3\alpha\pi)^2 t}\right]\sin(3\pi x)$$