Day 10

Continue eigenfunction expansions but introduce them as integral transforms...

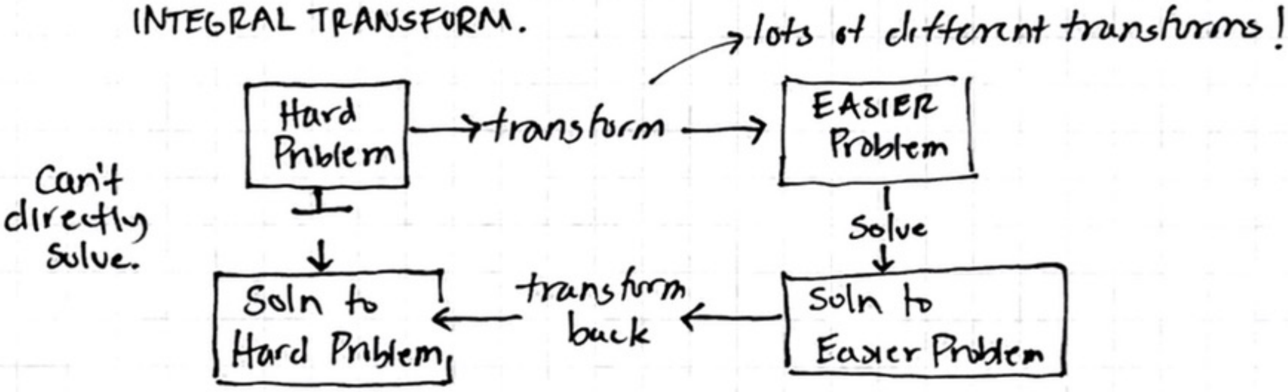
The general form $$F(s) = \int_A^B f(t)\, K(s,t)\, dt$$

$K(s,t)$ is the kernel of the transformation. We choose K so it...

\#1 Fits the boundary and initial conditions.

\#2 Simplifies the problem.

IDEA BEHIND INTEGRAL TRANSFORM.



(EE) Eigenfunction expansion = Integral transform with kernel of eigenfunctions.

Eigenfunct. Expand (EE) we assume: $u(x,t) = \sum_{n=1}^{\infty} T_n(t)\, \varphi_n(x)$ $\rightarrow$ eigenfunctions.

If we apply orthogonality FIRST and solve for T_n ...

$$\int_0^L u(x,t)\, \varphi_m(x)\, dx = \int_0^L \sum_{n=1}^{\infty} T_n\, \varphi_n(x)\, \varphi_m(x)\, dx$$

$$= \frac{T_n}{A}$$ where A is the orthogonality constant...

so $$T_n(t) = A\int_0^L u(x,t)\varphi_n(x)\,dx$$

Kernel.

To use this we plug each part of our problem into the integral and transform to a problem in terms of $T_n(t)$.

Ex $$\left.\begin{aligned} &u_t = \alpha^2 u_{xx} \\ &u(0,t) = 0 \\ &u(1,t) = t \\ &u(x,0) = \sin(\pi x) \end{aligned}\right\}$$ Homogeneous PDE nonhomogeneous BC's !

Transform BC's first... $u(x,t) = S(x,t) + v(x,t)$

where $S(x,t) = A(t)[1-x] + B(t)x$ and satisfies the BC's... $S(0,t)=0$ $S(1,t)=t$

$$S(0,t) = A(t) = 0 \qquad S(1,t) = B(t) = t$$

so $S(x,t) = tx$

$$\boxed{u(x,t) = tx + v(x,t).}$$

then $u_t = x + v_t$ $\quad u_{xx} = v_{xx}$ $\quad u(x,0) = v(x,0) = \sin(\pi x)$

NEW PDE $$\left.\begin{aligned} &v_t = \alpha^2 v_{xx} - x \\ &v(0,t) = 0 \\ &v(1,t) = 0 \\ &v(x,0) = \sin(\pi x) \end{aligned}\right\}$$ Nonhomogeneous PDE homogeneous BC's.

Now we can Eigenfunction Expand.

Use integral transform method:

$$v(x,t) = \sum_{n=1}^{\infty} T_n(t)\sin(n\pi x) \quad \text{-so-} \quad T_n(t) = 2\int_0^1 v(x,t)\sin(n\pi x)\,dx.$$

INVERSE TRANSFORM $\qquad$ TRANSFORM

$$\text{so} \quad T_n(t) = 2\int_0^1 \_\_\_ \,\sin(n\pi x)\,dx$$

put whole problem through the transform...

Consider: U_t

$$2\int_0^1 U_t \sin(n\pi x)\,dx = 2\int_0^1 \frac{d}{dt}U(x,t)\sin(n\pi x)\,dx$$

$$= \frac{d}{dt}\underbrace{\left[2\int_0^1 U(x,t)\sin(n\pi x)\,dx\right]}_{\text{this is the definition of the transform!}} = \frac{d}{dt}T_n(t) = T_n'(t)$$

Consider: U_{xx}

$$2\int_0^1 U_{xx}\sin(n\pi x)\,dx =$$ here we can't extract the derivative of x !!

Integration by parts !!

let

$u = \sin(n\pi x)$ $\quad dv = U_{xx}\,dx$

$du = n\pi\cos(n\pi x)$ $\quad v = U_x$

$$= 2\sin(n\pi x)U_x\Big|_0^1 \to 0 - (n\pi)2\int_0^1 U_x\cos(n\pi x)\,dx$$

let $u = \cos(n\pi x)$ $\quad dv = U_x\,dx$

$du = n\pi\sin(n\pi x)$ $\quad v = U$

$$= \cos(n\pi x)\,U\Big|_0^1 \to 0 - (n\pi)^2\cdot 2\int_0^1 U(x,t)\sin(n\pi x)\,dx \longrightarrow$$ This is the definition of the transform.

Here we used our BC's!

$$= -(n\pi)^2\,T_n(t)$$

Consider: $U(x,0) = \sin(\pi x)$

$$2\int_0^1 U(x,0)\sin(n\pi x)\,dx = 2\int_0^1 \sin(\pi x)\sin(n\pi x)\,dx$$

$$T_n(0) = \begin{cases} 1 & n=1 \\ 0 & \text{otherwise.} \end{cases}$$

Consider: $f(x,t) = x$ the nonhomogeneous part
YES we must transform this too !!

$$2\int_0^1 x\sin(n\pi x)\,dx = -\frac{2x}{n\pi}\cos(n\pi x)\Big|_0^1 + \frac{2}{n\pi}\int_0^1 \cos(n\pi x)\,dx$$

let $u = x$ $\quad dv = \sin(n\pi x)$

$du = 1$ $\quad v = -\frac{\cos(n\pi x)}{n\pi}$

$$= \frac{-2}{n\pi}\cos(n\pi) + \frac{2}{(n\pi)^2}\cancelto{0}{\sin(n\pi x)}\Big|_0^1$$

$$= \frac{-2}{n\pi}(-1)^{n} = \frac{2}{n\pi}(-1)^{n+1}$$

OUR PROBLEM IN TRANSFORM SPACE

$$\left.\begin{aligned} \underbrace{T_n'}_{u_t} &= \overbrace{-(\alpha n\pi)^2 T_n}^{\alpha^2 u_{xx}} + \overbrace{\frac{2}{n\pi}(-1)^{n}}^{-x} \\ T_n(0) &= \begin{cases} 1 & n=1 \\ 0 & \text{otherwise} \end{cases} \end{aligned}\right\}$$

This problem should NOT have any x's

↓

we transformed these out of the problem!

Now we solve the infinite # of ode's.

How many cases? $n=1$, $n=$ even, $n=$ odd.

only different by ± 1...

$n=1$

$$\left.\begin{aligned} T_1' &= -(\alpha\pi)^2 T_1 + \frac{2}{\pi} \\ T_1(0) &= 1 \end{aligned}\right\}$$

$T_1 = T_{1_H} + T_{1_{NH}}$

$T_{1_H} = Ae^{-(\alpha\pi)^2 t}$

$T_{1_{NH}} = C$ plug in...

$0 = -(\alpha\pi)^2 C + \frac{2}{\pi}$

so $C = \frac{-2}{\alpha^2\pi^3}$

general $T_1 = Ae^{-(\alpha\pi)^2 t} + \frac{2}{\alpha^2\pi^3}$

apply the IC $T_1(0) = A + \frac{2}{\alpha^2\pi^3} = 1$

$$A = 1 - \frac{2}{\alpha^2\pi^3}$$

$$T_1(t) = \frac{-2}{\alpha^2\pi^3}\left[\left(\frac{\alpha^2\pi^3}{2} + 1\right)e^{-(\alpha\pi)^2 t} - 1\right]$$

n=2

$$\left.\begin{aligned} T_2' &= -(\alpha 2\pi)^2 T_2 - \frac{2}{2\pi} \\ T_2(0) &= 0 \end{aligned}\right\} \quad T_2 = T_{2_H} + T_{2_{NH}}$$

$$T_{2_H} = Ae^{-(2\pi\alpha)^2 t}$$

$$T_{2_{NH}} = C \;\ldots\; -(2\pi\alpha)^2 C - \frac{2}{\pi} = 0$$

general $T_2(t) = Ae^{-(2\pi\alpha)^2 t} - \dfrac{2}{2(2\alpha)^2\pi^3}$ $\qquad C = -\dfrac{2}{2(2\alpha)^2\pi^3}$

$$T_2(0) = A - \frac{2}{(\alpha)^2\pi^3(2)^3} = 0 \qquad A = \frac{2}{(\alpha)^2\pi^3(2)^3}$$

$$T_2(t) = \frac{-2}{\alpha^2(2\pi)^3}\left[e^{-(2\pi\alpha)^2 t} - 1\right]$$

n=3 We should see a pattern here...

$e^{-(n\alpha\pi)^2 t}$ in general.

$$\left.\begin{aligned} T_3' &= -(3\alpha\pi)^2 T_3 + \frac{2}{3\pi} \\ T_3(0) &= 0 \end{aligned}\right\} \quad T_3 = T_{3_H} + T_{3_{NH}}$$

$$T_{3_H} = Ae^{-(3\alpha\pi)^2 t}$$

$$T_{3_{NH}} = C \ldots \quad -(3\alpha\pi)^2 C + \frac{2}{3\pi} = 0$$

$$T_3(t) = Ae^{-(3\pi\alpha)^2 t} + \frac{2}{\alpha^2(3\pi)^3} \qquad C = \frac{2}{\alpha^2(3\pi)^3}$$

$$T_3(0) = A + \frac{2}{\alpha^2(3\pi)^3} = 0 \qquad A = \frac{-2}{\alpha^2(3\pi)^3}$$

$$A = \frac{2(-1)^n}{\alpha^2(n\pi)^3} \qquad C = \frac{2(-1)^{n+1}}{\alpha^2(n\pi)^3}$$

$$T_3(t) = \frac{+2}{\alpha^2(3\pi)^3}\left[e^{-(3\alpha\pi)^2 t} - 1\right]$$

So the general soln for $n>2$ is given by...

$$T_n(t) = \frac{2(-1)^{n+1}}{\alpha^2(n\pi)^3}\left[e^{-(n\pi\alpha)^2 t} - 1\right]$$

Now write down our solution "Inverse Transform"

$$\mathcal{U}(x,t) = \sum_{n=1}^{\infty} T_n(t) \sin(m\pi x) \qquad \text{so} \ldots$$

$$\mathcal{U}(x,t) = \frac{-2\sin(\pi x)}{\alpha^2\pi^3}\left[\left(\frac{\alpha^2\pi^3}{2}+1\right)e^{-(\alpha\pi)^2 t} - 1\right] + \sum_{n=2}^{\infty} \frac{2(-1)^{n+1}}{\alpha^2(n\pi)^3}\left[e^{-(n\pi\alpha)^2 t} - 1\right]\sin(n\pi x)$$

Technicaley some of this can be written

$$= -e^{-(\alpha\pi)^2 t}\sin(\pi x) + \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{\alpha^2(n\pi)^3}\left[e^{-(n\pi\alpha)^2 t} - 1\right]\sin(n\pi x)$$

AND our final solution is...

$$\boxed{u(x,t) = xt - e^{-(\alpha\pi)^2 t}\sin(\pi x) + \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{\alpha^2(n\pi)^3}\left[e^{-(n\pi\alpha)^2 t} - 1\right]\sin(n\pi x).}$$

There are lots of different transforms!

Different choices for $K(s,t)$...

Finite Domain $\longrightarrow$ Finite Transform
Sum as the inverse.

(semi) Infinite Domain $\longrightarrow$ (semi) Infinite Transform
Laplace, Fourier, etc...