DAY 12. NOTES.

Sine and Cosine Transforms... today we start working more generally on transform methods.

Define Transforms:

$$\mathcal{F}_s[f] = F(\omega) = \frac{2}{\pi}\int_0^\infty f(t)\sin(\omega t)\,dt \qquad \text{Fourier Sine}$$

$$\mathcal{F}_s^{-1}[F] = f(t) = \int_0^\infty F(\omega)\sin(\omega t)\,d\omega \qquad \text{Inverse Sine}$$

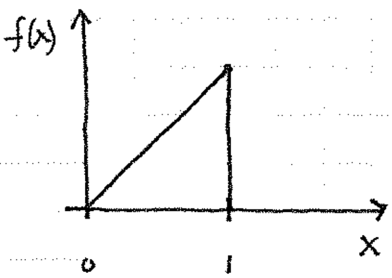
$$\mathcal{F}_c^{-1}[f] = F(\omega) = \frac{2}{\pi}\int_0^\infty f(t)\cos(\omega t)\,dt \qquad \text{Fourier Cosine}$$

$$\mathcal{F}_c^{-1}[F] = f(t) = \int_0^\infty F(\omega)\cos(\omega t)\,d\omega \qquad \text{Inverse Cosine.}$$

NOTICE that these allow us to deal with semi-infinite problem in x, or t. Also we can now deal with on/off functions.

Ex

$$f(x) = \begin{cases} x & 0 < x < 1 \\ 0 & x > 1 \end{cases}$$



Find the sine transform.

- imagine all the real life situations that deal with on/off forcing functions... Earthquakes, Temperature sources, ...

just do the integral...

$$\frac{2}{\pi}\int_0^\infty f(x)\sin(\omega x)\,dx = \frac{2}{\pi}\int_0^1 x\sin(\omega x)\,dx$$

notice that I changed from t to x.

let $u = x$ $\quad dv = \sin(\omega x)$

$du = dx$ $\quad v = \frac{-1}{\omega}\cos(\omega x)$

$$= \frac{2}{\pi}\left[-\frac{x}{\omega}\cos(\omega x)\Big|_0^1 + \int_0^1 \frac{1}{\omega}\cos(\omega x)\,dx\right]$$

$$= \frac{-2}{\pi\omega}\cos(\omega) + \frac{1}{\omega^2}\sin(\omega x)\Big|_0^1 = \boxed{\frac{-2}{\pi\omega}\cos(\omega) + \frac{1}{\omega^2}\sin(\omega x)}$$

There should be no x's after the transform.

NOTE... often when doing transforms it is easier to look up the inverse on the table p. 399.

Lets use this to solve a PDE.

$$\left.\begin{aligned} u_t &= u_{xx} \\ u(0,t) &= A \qquad A \text{ is a constant} \\ u(x,0) &= 0 \\ & \qquad\qquad 0 < x < \infty \end{aligned}\right\}$$ semi infinite rod with a constant temperature at $x=0$

We also assume $\lim_{x\to\infty} |u| = 0$ basically u must go to zero as $x \to \infty$

Here, and whenever doing transforms, we don't care whether or not we have homogeneous BCs. We will use them in our transform.

$\mathcal{F}_s[f] = F(\omega) = \frac{2}{\pi}\int_0^\infty f(x)\sin(\omega x)\,dx$ transform all parts.

NOTE... sine transform when u is specified on boundary
cosine transform when u_x is specified.

$$\mathcal{F}_s[u_t] = \frac{d}{dt}F = F'$$

$$\mathcal{F}_s[u_{xx}] = \frac{2}{\pi}\int_0^\infty u_{xx}\sin(\omega x)\,dx$$

integration by parts twice.

TABLE METHOD...

u	dv	$+/-$
$\sin(\omega x)$	u_{xx}	$+$
$\omega\cos(\omega x)$	u_x	$-$
$-\omega^2\sin(\omega x)$	u	$+$

$$= \frac{2}{\pi}u_x \overset{0}{\cancel{\sin}}(\omega x)\Big|_0^\infty - \frac{2}{\pi}\omega\cos(\omega x)\,u\Big|_0^\infty - \frac{2}{\pi}\omega^2\int_0^\infty u\sin(\omega x)\,dx$$

$u_x \to 0$ as $x \to \infty$

$$= +\frac{2\omega}{\pi}\overset{1}{\cancel{\cos(0)}}\,\overset{A}{\cancel{u(0,t)}} - \omega^2 F$$

after applying limits of integration and using BC and conditions at $x\to\infty$

$$= +\frac{2\omega A}{\pi} - \omega^2 F$$

$$\mathcal{F}_s\left[u(x,0)\right] = F(0) = \int_0^\infty \emptyset \, dx = 0 \qquad F(0) = 0$$

NEW PROBLEM $\left.\begin{array}{l} F' = -w^2 F + \frac{2wA}{\pi} \\ F(0) = 0 \end{array}\right\}$ use integrating factor.

$$\mu = e^{\int w^2 dt} = e^{w^2 t} \rightarrow \frac{d}{dt}\left[Fe^{w^2t}\right] = \frac{2wA}{\pi} e^{w^2 t}$$

$$\rightarrow Fe^{w^2t} = \frac{2wA}{\pi w^2} e^{w^2 t} + C$$

$$F = \frac{2A}{\pi w} + Ce^{-w^2 t} \qquad F(0) = \frac{2A}{\pi w} + c = 0$$

$$C = \frac{-2A}{\pi w}$$

$$F(t) = \frac{2A}{\pi w}\left[1 - e^{-w^2 t}\right]$$

Now do an inverse transform to get the final soln...

$$\mathcal{F}_s^{-1}[F] = \int_0^\infty \frac{2A}{\pi w}\left[1 - e^{-w^2 t}\right] \sin(wt)\, dw$$

look this up on table ... easier than evaluating the integral.

p402 #11

$$u(x,t) = A\,\mathrm{erfc}\left[x/2\sqrt{t}\right]$$

This is our soln. The "A" is is the constant from our Boundary Conditions.

GAUSS ERROR FUNCTION

$$\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dx \qquad \mathrm{erfc} = 1 - \mathrm{erf}(x) = \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-t^2} dt.$$

originally these were defined in measurement theory and actually used to measure error... however they appear in lots of places where they have nothing to do with error.