Day 14

Farlow Lesson 13

The Laplace Transform ... $\mathcal{L}[f] = \int_0^\infty f(t)e^{-st}\,dt$

- a wider class of functions can be transformed by the Laplace Transform ... e^{-st} acts as a damping term where we need to specify $s > a$ such that the integral converges.

Transforms of Partial Derivatives:

NOTE: Because time varies $0 < t < \infty$ we will usially transform out the time dependence.

$$\mathcal{L}[u_t] = sU(x,s) - u(x,0)$$

$$\mathcal{L}[u_{tt}] = s^2U(x,s) - su(x,0) - u_t(x,0)$$

these should look familiar from ODE's ... look up the Laplace transform chapter in Edwards + Penney.

$$\mathcal{L}[u_x] = \frac{\partial U}{\partial x}$$

$$\mathcal{L}[u_{xx}] = \frac{\partial^2 U}{\partial x^2}$$

→ since we are transforming time the spacial derivatives pop out front!

FINITE CONVOLUTION

$$f * g = \int_0^t f(\tau)\,g(t-\tau)\,d\tau = \int_0^t f(t-\tau)\,g(t)\,d\tau$$

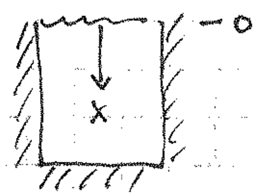
We use the property $\mathcal{L}[f * g] = \mathcal{L}[f]\,\mathcal{L}[g]$

or more specifically when finding the inverse

$$f * g = \mathcal{L}^{-1}(\mathcal{L}[f]\,\mathcal{L}[g])$$

Ex Heat Conduction in a Semi Infinite Medium.

$$\begin{cases} u_t = u_{xx} & 0 < x < \infty \quad 0 < t < \infty \\ u_x\big|_{x=0} - u\big|_{x=0} = 0 & 0 < t < \infty \\ u(x,0) = u_0 & 0 < x < \infty \end{cases}$$



Here our goal is to transform the time variable using a Laplace transform.

* we could transform space instead if we wanted ...

Apply $\mathcal{L}[(*)] = \int_0^\infty (*) e^{-st} dt$ to the PDE

LHS ... $\mathcal{L}[u_t] = sU - u(x,0)$ where $U = \int_0^\infty u e^{-st} dt = \mathcal{L}[u]$

here we use our IC so we don't later need to transform it!

$\mathcal{L}[u_t] = sU - u_0$

RHS ... $\mathcal{L}[u_{xx}] = U_{xx} \rightsquigarrow$ spacial derivative pops out in front of integral

NEW PDE: $\mathcal{L}[u_t] = \mathcal{L}[u_{xx}]$

$sU - u_0 = U_{xx}$ or $U_{xx} - sU = -u_0$

second order constant coef. non homogeneous ODE

WE NEED BC's ... TRANSFORM THE BC

$\mathcal{L}[u_x - u] = \mathcal{L}[0] \qquad U_x - U = 0$ or $U_x(0) = U(0)$

Now solve $\begin{cases} U_{xx} - sU = -u_0 \\ U_x(0) = U(0) \end{cases}$

general soln = homogeneous + particular

Homogeneous...

Characteristic $r^2 - s = 0 \quad r = \pm\sqrt{s}$

$$V_H = Ae^{-\sqrt{s}x} + Be^{\sqrt{s}x}$$

Particular

Guess something that matches the nonhomogeneous part...

$V_p = C = \text{constant}$ This should include your guess and all of its derivatives... derivative of a constant $= 0$

check for duplication... no duplication of V_H

plug into ODE $V_{p_{xx}} = 0 \quad 0 - sC = -u_0 \Rightarrow C = +\frac{u_0}{s}$

General Soln: $V = Ae^{-\sqrt{s}x} + Be^{\sqrt{s}x} + \frac{u_0}{s}$

Boundary Conditions...

First we assume V is bounded as $x \to \infty$
so $B = 0$ since $e^{\sqrt{s}x}$ would grow to ∞ as $x \to \infty$

$$V_x = -A\sqrt{s}\, e^{-\sqrt{s}x}$$

$$\underbrace{V_x(0) = -A\sqrt{s} = A + \frac{u_0}{s}}_{\text{solve for } A} = V(0)$$

$$-\frac{u_0}{s} = (1+\sqrt{s})A \qquad A = -\frac{u_0}{s(1+\sqrt{s})}$$

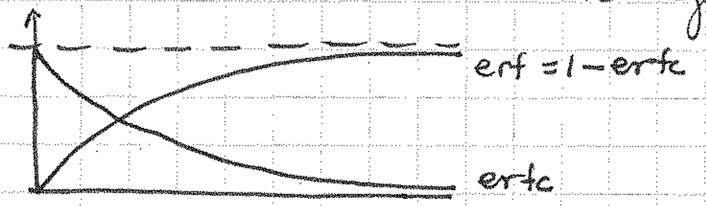
Final Soln for V... $V(x) = -u_0\left[\frac{e^{-\sqrt{s}x}}{s(1+\sqrt{s})}\right] + \frac{u_0}{s}$

now we need the inverse transform to get back to our solution.

$$u(x,t) = \mathcal{L}^{-1}[V] = -u_0\,\mathcal{L}^{-1}\left[\frac{e^{-\sqrt{s}x}}{s(1+\sqrt{s})}\right] + u_0\,\mathcal{L}^{-1}\left[1/s\right]$$

$$u(x,t) = u_0 - u_0\left[\operatorname{erfc}\left(x/2\sqrt{t}\right) - \operatorname{erfc}\left(\sqrt{t} + x/2\sqrt{t}\right)e^{(x+t)}\right]$$

use maple or another computer program to get this.



$$\operatorname{erfc}(x) = \frac{2}{\pi}\int_x^{\infty} e^{-\xi^2}\, d\xi$$

what happens as $t \to \infty$ for different x values...

$x=0$ @ the top of the container $\quad$ $\operatorname{erfc}(0) = 1$

so as $t \to \infty$ $\quad$ $\operatorname{erfc}(t \to \infty) = 0$

faster than e^t

so we get $u_0 - u_0 \to 0$

$x \to \infty$ @ bottom of container $\quad$ $\operatorname{erfc}(\infty) = 0$

so $\quad u = u_0$ stays warm.