PDEs DAY 16.

Moving Forward — the plan is to do through Lesson 30
if time we will derive some eqns from physical systems.

11/6 Midterm — During Lab
Study Guide — next time
Review Session 11/4 evening? 5pm?
Open Book.

11/7 NO Class.

Today Lesson 15 — The Convection Term.

u_{xx} → Diffusion or mixing or disapating
u_x → Convection or being carried by a current.

$u_t = Du_{xx} - Vu_x$ Diffusion - Convection Eqn.

Start by considering the easier case $D=0$, Pure Convection.
• Understand a Simpler problem to help solve a more complicated problem!

PURE CONVECTION

$$\left.\begin{aligned} u_t &= -Vu_x \\ u(0,t) &= P \\ u(x,0) &= 0 \end{aligned}\right\} \quad \begin{aligned} &0 < x < \infty \\ &\text{semi infinite} \end{aligned}$$

start with zero concentration then add substance as an input at $x=0$

- What might we expect?... substance in at $x=0$
pushed along by convection.
does not dissapate.

Use Laplace Transform ... semi-infinite domain

$$\mathcal{L}[u_t] = sU - \cancelto{0}{u(x,0)} = sU \qquad \mathcal{L}[u_x] = U_x \qquad \mathcal{L}[P] = P/s$$

NEW PROBLEM: $$\left.\begin{aligned} sU &= -vU_x \\ U(0) &= P/s \end{aligned}\right\} \quad \text{so } U(x,s) = \frac{P}{s}e^{-s/v\,x}$$

now transform back.

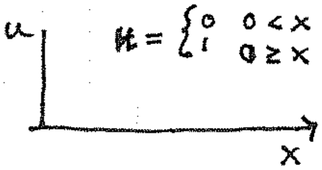
$$u(x,t) = \mathcal{L}^{-1}\left[\frac{P}{s} e^{-\frac{s}{V}x}\right] = P\,H(t - x/V) = \begin{cases} 0 & t < x/V \\ 1 & t \geq x/V \end{cases}$$

p 406 #11 $H(t-a) \longleftrightarrow \frac{e^{-as}}{s}$

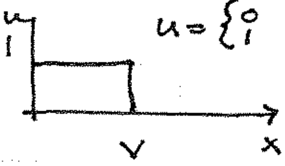
$$H(\xi) = \begin{cases} 0 & \xi < 0 \\ 1 & \xi \geq 0 \end{cases}$$

This is a square wave traveling with velocity V.

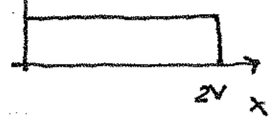
$t=0$ $u = \begin{cases} 0 & 0 < x \\ 1 & 0 \geq x \end{cases}$



$t=1$ $u = \begin{cases} 0 & V < x \\ 1 & V \geq x \end{cases}$



$t=2$ $u = \begin{cases} 0 & 2V < x \\ 1 & 2V \geq x \end{cases}$



- Like a conveyer belt moving the substance down stream at a velocity V.

<u>NOW ADD DIFFUSION</u>

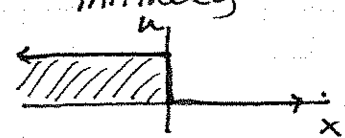
$$u_t = D u_{xx} - V u_x \qquad -\infty < x < \infty$$ infinite domain

so diffusion is not complicated @ $x=0$.

$$u(x,0) = 1 - H(x)$$

- what do we expect?

initially



<u>NOTE</u>
This problem can be solved by straight transform

WATCHING FROM SIDE $\rightarrow$ The conveyer belt will move with velocity V and the substance will diffuse.

WHAT IF WE WERE SITTING ON THE BELT $\longrightarrow$ We would only see diffusion, a problem we have solved many times!

CHANGE OF COORDINATES! $\quad \xi = x - Vt \quad$ attaches us to the conveyor belt.
change of independent variables $\rightarrow$ $\tau = t$

RECALL The <u>Chain Rule</u>

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial \xi}\overbrace{\frac{\partial \xi}{\partial x}}^{1} + \frac{\partial u}{\partial \tau}\overbrace{\frac{\partial \tau}{\partial x}}^{0} = \frac{\partial u}{\partial \xi}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left[\frac{\partial u}{\partial x}\right] = \frac{\partial}{\partial \xi}\left[\frac{\partial u}{\partial \xi}\right]\overbrace{\frac{\partial \xi}{\partial x}}^{1} + \frac{\partial}{\partial \tau}\left[\frac{\partial u}{\partial \xi}\right]\overbrace{\frac{\partial \tau}{\partial x}}^{0} = \frac{\partial^2 u}{\partial \xi^2}$$

$$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial \xi}\overbrace{\frac{\partial \xi}{\partial t}}^{-V} + \frac{\partial u}{\partial \tau}\overbrace{\frac{\partial \tau}{\partial t}}^{1} = -V\frac{\partial u}{\partial \xi} + \frac{\partial u}{\partial \tau}$$

initial condition $\xi = x - V(0)$, $\tau = 0$ $\longrightarrow$ $u(\xi, 0) = 1 - H(\xi)$

NEW PROBLEM: $-\cancel{V u_\xi} + u_\tau = D u_{\xi\xi} - \cancel{V u_\xi}$

$$\left.\begin{array}{ll} u_\tau = D u_{\xi\xi} & -\infty < \xi < \infty \\ u(\xi, 0) = 1 - H(\xi) = \phi(\xi) & 0 < \tau < \infty \end{array}\right\}$$ Diffusion on infinite domain.

What method? Fourier Transform!

we actually solved this before... but lets do it again.

$$\mathcal{F}(\omega, \tau) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u \, e^{-i\omega\xi} \, d\xi$$

Transform from ξ to ω in space ... ξ is our space var.

$$\mathcal{F}[u_\tau] = V_\tau \quad \mathcal{F}[u_{\xi\xi}] = -\omega^2 V \quad \mathcal{F}[\phi(\xi)] = \Phi(\omega)$$

NOTE... in the end we transform back so I do the transform IC only symbolically

NEW PROBLEM: $\left.\begin{array}{l} V_\tau = -D\omega^2 V \\ V(0) = \Phi(\omega) \end{array}\right\}$

$$V(\omega, \tau) = \Phi(\omega) e^{-D\omega^2\tau}$$

undo the transform: $\mathcal{F}^{-1}\left[\Phi(\omega) e^{-D\omega^2\tau}\right]$

product $\Rightarrow$ convolution.

$$u(\xi, \tau) = \mathcal{F}^{-1}[\Phi(\omega)] * \mathcal{F}^{-1}\left[e^{-D\omega^2\tau}\right] = (1 - H(\xi)) * \frac{1}{\sqrt{2D\tau}} e^{-\xi^2/4D\tau}$$

p. 400 #6

with $\frac{1}{4a^2} = D\tau$

$a = \frac{1}{2\sqrt{D\tau}}$ + balance constant!

$$u(\xi, \tau) = \frac{1}{2\sqrt{\pi D\tau}} \int_{-\infty}^{\infty} (1 - H(\xi)) \, e^{-(\xi-\beta)^2/4D\tau} \, d\beta$$

β → dummy var for integration.

$$= \frac{1}{2\sqrt{\pi D\tau}} \int_{-\infty}^{0} e^{-(\xi-\beta)^2/4D\tau} \, d\beta$$

let $w = \frac{\xi - \beta}{\sqrt{4D\tau}}$ then $dw = \frac{-1}{\sqrt{4D\tau}} d\beta$

when $\beta = -\infty \quad w = \infty$

$\beta = 0 \quad w = \frac{\xi}{\sqrt{4D\tau}}$

$$u(\xi,\tau) = \frac{1}{\sqrt{\pi}}\int_{\xi/2\sqrt{D\tau}}^{\infty} e^{-w^2}\,dw = \begin{cases} \frac{1}{2}\left[1 + \mathrm{erf}\left(\frac{-\xi}{2\sqrt{D\tau}}\right)\right] & \xi < 0 \\ \frac{1}{2}\,\mathrm{erfc}\left(\frac{\xi}{2\sqrt{D\tau}}\right) & \xi \ge 0 \end{cases}$$

MATLAB infdiff(time)

we see the initial wave diffusing!

Are we done? NO!! still need to transform back to the original coordinates.

$t = \tau$
$x = \xi + Vt$

$$u(x,t) = \begin{cases} \frac{1}{2}\left[1 + \mathrm{erf}\left(\frac{Vt - x}{2\sqrt{Dt}}\right)\right] & Vt > x \\ \frac{1}{2}\,\mathrm{erfc}\left(\frac{x - Vt}{2\sqrt{Dt}}\right) & Vt \le x \end{cases}$$

MATLAB infconv (time)

we see the leading edge more downstream <u>and</u> break up as it goes along.

MOST IMPORTANT TODAY: <u>CHANGE OF COORDINATES</u>!
(change of variables)

NOTES ... Lots of ways to approach problems like this!

FROM LESSON 8 $\longrightarrow$

$$u(x,t) = e^{v(x - vt/2)/2\alpha^2}\, w(x,t)$$

transform the dependent variable!

$V = v$ and $\alpha^2 = D$

$$\left.\begin{aligned} u_t &= D u_{xx} - v u_x \\ u(x,0) &= 1 - H(x) \end{aligned}\right\} \longrightarrow \begin{aligned} w_t &= D w_{xx} \\ w(x,0) &= e^{-x/2D}(1 - H(x)) \end{aligned}$$

then do a Fourier transform.

FROM LESSON 12 $\longrightarrow$ just do the Fourier Transform in space $x \to \omega$

$$\left.\begin{aligned} u_t &= D u_{xx} - v u_x \\ u(x,0) &= 1 - H(x) \end{aligned}\right\} \longrightarrow \begin{aligned} U_t &= -D\omega^2 U - i\omega U \\ U(0) &= \Phi(\omega) \end{aligned}$$

solve and find the inverse.

FROM LESSON 13 $\longrightarrow$ just do a Laplace Transform in time $t \to s$

$$\left.\begin{aligned} u_t &= D u_{xx} - v u_x \\ u(x,0) &= 1 - H(x) \end{aligned}\right\} \longrightarrow sU - 1 + H(x) = D U_{xx} - v U_x$$

$U \to 0$ as $|x| \to \pm\infty$ bounded.

NOT SO PRETTY !!

CAN'T SEPARATE VARIABLES!
NO EIGENFUNCTION EXPAND!

Because of infinite or semi infinite domain.

Partial Differential Equations

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Homework and Reading - Day 16:

Due Thursday 10/31

Read and Take Notes

1. *Farlow* - Lesson 15
2. *Farlow - read ahead* - You can read ahead of the class if you want. The next chapters we will be covering are Lessons 16 and 19.

Homework

1. *Problems* - Do problems all problems at the end of Chapter 15.
2. *Review* - Solve the following PDEs using the method of your choice. First discuss what physical system is being represented and what you expect to happen in the long term solution. Then choose a solution method and state why you picked the method that you picked and why the other methods are less suitable. After solving, does your solution match your expectations?

$$\begin{aligned} u_t &= u_{xx} + 1 \\ u(0,t) &= 0 \\ u(1,t) &= 0 \\ u(x,0) &= 0 \end{aligned} \tag{1}$$

$$\begin{aligned} u_t &= u_{xx} \quad 0 < x < \infty \\ u(0,t) &= 0 \\ u(x,0) &= e^{-x} \end{aligned} \tag{2}$$