A cool consequence of Laplace Transform <u>Duhamel's Principle</u>

- General Idea → solve a hard problem
  by understanding a much easier one.

Consider the PDE:

$$\left.\begin{aligned} u_t &= u_{xx} \\ u(0,t) &= 0 \\ u(1,t) &= f(t) \\ u(x,0) &= 0 \end{aligned}\right\}$$

Why is this a "<u>Hard</u>" problem?

Time varying boundary condition!

Instead we consider the problem:

$$\left.\begin{aligned} w_t &= w_{xx} \\ w(0,t) &= 0 \\ w(1,t) &= 1 \\ w(x,0) &= 0 \end{aligned}\right\}$$

constant <u>Boundaries</u>...

SOLVE THESE SIDE BY SIDE ...

where do things go crazy?

1. LAPLACE TRANSFORM

$$sW - \cancel{w(x,0)}^{0} = W''$$

$$\left.\begin{aligned} W'' - sW &= 0 \\ W(0) &= 0 \\ W(1) &= \mathcal{L}\{1\} = 1/s \end{aligned}\right\}$$

similarly

$$\left.\begin{aligned} V'' - sV &= 0 \\ V(0) &= 0 \\ V(1) &= \mathcal{L}\{f(t)\} = F(s) \end{aligned}\right\}$$

These ODEs will have similar solns

$$W(x) = Ae^{\sqrt{s}x} + Be^{-\sqrt{s}x}$$

$$W(0) = A + B = 0 \quad A = -B$$

$$W(x) = A\left(e^{\sqrt{s}x} - e^{-\sqrt{s}x}\right) = A\sinh(x\sqrt{s})$$

$$V = A\sinh(x\sqrt{s})$$

Apply the second condition

$$W(1) = A\sinh(\sqrt{s}) = 1/s$$

$$A = \frac{1}{s}\cdot\frac{1}{\sinh(\sqrt{s})}$$

$$W(x,s) = \frac{1}{s}\,\frac{\sinh(x\sqrt{s})}{\sinh(\sqrt{s})}$$

$$V(1) = A\sinh(\sqrt{s}) = F(s)$$

$$A = \frac{F(s)}{\sinh(\sqrt{s})}$$

$$V(x,s) = F(s)\,\frac{\sinh(x\sqrt{s})}{\sinh(\sqrt{s})}$$

The final step is to invert ...

$$w(x,t) = x + \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{(-1)^n}{n} e^{-(n\pi)^2 t} \sin(n\pi x)$$

cheat! We actually know this solution from separation of variables

This is tricky!

mult and divide by s: This is W

$$F(s)\, s\, \frac{\sinh(x\sqrt{s})}{s\sinh(\sqrt{s})} = V$$

$V = F(s) \cdot s \cdot W$

~~Why... we will recognize this as RULE #8.~~

First recognize w as $\mathcal{L}^{-1}\left[\frac{\sinh(x\sqrt{s})}{s\sinh(\sqrt{s})}\right]$

use this

then $\mathcal{L}[w_t] = s \cdot W - w(x,0)$ (*)

IDENTITY

where $w(x,0) = 0$ From pde.

$V = F(s)\, s \cdot W$ by * above and $V = F(s)\,\mathcal{L}[w_t]$ by identity.

Lets invert this...

so $\mathcal{L}^{-1}\{V\} = \mathcal{L}^{-1}\{F(s)\cdot\mathcal{L}[w_t]\} = \mathcal{L}^{-1}\{F(s)\} * w_t$ * convolution.

*BC.

$= f(t) * w_t$. by the identity.

soln to the hard problem...

$$u(x,t) = \int_0^t f(\tau)\, w_\tau(x, t-\tau)\, d\tau$$ plugging into convolution.

or integrating by parts, since we can't do term by term differentiation of our series $w(x,t)$

This gives a cool result.

let $u = f$ $\quad dv = w_\tau$

$du = f'$ $\quad v = w\,(-1) = -w$

$$u(x,t) = -f(\tau)\, w(x,t-\tau)\Big|_0^t + \int_0^t f'(\tau)\, w(x,t-\tau)\, d\tau$$

$$= -f(t)\, \cancelto{0}{w(x,0)} + f(0)\, w(x,t) + \int_0^t f'(\tau)\, w(x,t-\tau)\, d\tau$$

$$u(x,t) = \int_0^t f'(\tau)\, w(x,t-\tau)\, d\tau + f(0)\, w(x,t)$$

This is in terms of $w(x,t)$ the soln to the easy problem!

In other words... no matter what our time varying BC's are we can sub them ~~this~~ into this integral and express our solution in terms of the easier problem!

Experimentally ... what does this mean?

If we can measure the solution to the problem even once ... then we can numerically solve the integral for any conditions on the boundary.

Ex 1-D rod that will be use in a wide variety of settings...

In the lab we measure the case of constant BCs and store that data. $w(x,t)$

Later we want to know how the rod will respond to new conditions ... we don't need to re-do the experiment.

$$u(x,t) = \int_0^t f'(\tau)\, w(x, t-\tau)\, d\tau + f(0)\, w(x,t)$$

we know both the original soln $w(x,t)$ and the new conditions.