- I will post HW solution over weekend.
  Go through and re-do / practice these problems.
- Example exam problems posted — Please Study First !!
  Then try the exam !!

Lesson 17 D'Alembert Solution of the Wave Eqn.

Today we start considering

$$u_{tt} = c^2 u_{xx} \qquad -\infty < x < \infty$$

$$\left.\begin{aligned} u(x,0) &= f(x) \\ u_t(x,0) &= g(x) \end{aligned}\right\} \text{ initial conditions.}$$

describes the motion of an infinite string with some initial condition.

For this problem: Laplace Transform in Time OR Fourier Transform in space would work.

Laplace: $\mathcal{L}[u] = \int_0^\infty u e^{-st} dt$

$$\mathcal{L}[u_{tt}] = s^2 U - s u(x,0) - u_t(x,0)$$
$$= s^2 U - s f(x) - g(x)$$
$$\mathcal{L}[u_{xx}] = U''$$

$$\left.\begin{aligned} c^2 U'' - s^2 U &= -s f(x) - g(x) \\ U \to 0 \text{ as } &|x| \to \infty \end{aligned}\right\}$$

NOTE: The bounded condition usually means only the non-homogeneous soln is needed.

Fourier: $\mathcal{F}[u] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u e^{-i\xi x} dx$

$$\mathcal{F}[u_{tt}] = U''$$

$$\mathcal{F}[u_{xx}] = -\xi^2 U$$

$$\left.\begin{aligned} U'' &= -c^2 \xi^2 U \\ U(0) &= \mathcal{F}[f(x)] = F(\xi) \\ U'(0) &= \mathcal{F}[g(x)] = G(\xi) \end{aligned}\right\}$$

Another way $\longrightarrow$ COORDINATE TRANSFORM "canonical coordinates"

see this in Classical and Quantum

sets of coordinates w/ time and space dependence that satisfy certain vector field properties.

let $\left.\begin{aligned} \xi &= x + ct \\ \eta &= x - ct \end{aligned}\right\}$ Like attaching to the "wave" going in both directions.

Now change variables in the PDE.

$$u(x,t) = \phi(x-ct) + \psi(x+ct)$$

all of this makes physical sense!

- this is actually a general solution to the wave equation!
- the sum of two moving waves going in opposite directions.
- the exact form of this soln depends on applying the initial conditions.

Apply the Initial Conditions:

$$u(x,0) = \boxed{\phi(x) + \psi(x) = f(x)} \quad (\star)$$

$$u_t(x,0) = \boxed{-c\phi'(x) + c\psi'(x) = g(x)}$$

integrate ~~wrt~~ wrt x... from x_0 to x

note $\phi'(x)$...

$$\frac{d}{dt}\phi(x-ct)\Big|_{t=0} = \frac{d\phi}{d\eta}\frac{d}{dt}(x-ct)\Big|_{t=0} = -c\frac{d\phi}{d\eta}\Big|_{t=0} = -c\frac{d\phi}{dx}$$

when $t=0$ $\eta = x$

$$\int_{x_0}^{x} -c\phi'(\xi) + c\psi'(\xi)\,d\xi = \int_{x_0}^{x} g(\xi)\,d\xi$$

$$\Big[-c\phi(\xi) + c\psi(\xi)\Big]\Big|_{x_0}^{x} = -c\phi(x) + c\psi(x) - \underbrace{K}_{-c\phi(x_0)+c\psi(x_0)} = \int_{x_0}^{x} g(\xi)\,d\xi.$$

$$\boxed{-c\phi(x) + c\psi(x) = \int_{x_0}^{x} g(\xi)\,d\xi + K.} \quad (\star)$$

$(\star)$ — two equations for two unknowns.

so $\phi(x) = f(x) - \psi(x)$

AND $\psi(x) = \phi(x) + \frac{1}{c}\int_{x_0}^{x} g(\xi)\,d\xi + K/c$

so $\phi(x) = f(x) - \phi(x) - \frac{1}{c}\int_{x_0}^{x} g(\xi)\,d\xi - K/c$

OR

$$\phi(x) = \frac{f(x)}{2} - \frac{1}{2c}\int_{x_0}^{x} g(\xi)\,d\xi - K/2c$$

$$\frac{du}{dx} = \frac{du}{d\xi}\frac{d\xi}{dx} + \frac{du}{d\eta}\frac{d\eta}{d\xi} = \frac{du}{d\xi} + \frac{du}{d\eta} = u_\xi + u_\eta$$

$$\frac{du}{dt} = \frac{du}{d\xi}\frac{d\xi}{dt} + \frac{du}{d\eta}\frac{d\eta}{dt} = c\frac{du}{d\xi} - c\frac{du}{d\eta} = c(u_\xi - u_\eta)$$

$$\frac{d^2u}{dx^2} = \frac{d}{dx}(\underbrace{u_\xi + u_\eta}_{\frac{du}{dx}\text{ from above}}) = \frac{d}{d\xi}(u_\xi + u_\eta)\frac{d\xi}{dx} + \frac{d}{d\eta}(u_\xi+u_\eta)\frac{d\eta}{dx}$$

$$= u_{\xi\xi} + u_{\eta\xi} + u_{\xi\eta} + u_{\eta\eta} = u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta} \quad (*)$$

mixed partials are equal.

$$\frac{d^2u}{dt^2} = \frac{d}{dt}c(u_\xi - u_\eta) = \frac{d}{d\xi}(c(u_\xi - u_\eta))\frac{d\xi}{dt} + \frac{d}{d\eta}(c(u_\xi - u_\eta))\frac{d\eta}{dt}$$

$$= c^2(u_{\xi\xi} - u_{\eta\eta}) - c^2(u_{\xi\eta} + u_{\eta\eta}) = c^2(u_{\xi\xi} - 2u_{\xi\eta} + u_{\eta\eta}) \quad (*)$$

plug these second derivatives into the pde:

$$c^2(u_{\xi\xi} - 2u_{\xi\eta} + u_{\eta\eta}) = c^2(u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta})$$

This simplifies to $\boxed{u_{\xi\eta} = 0}$ wow !! simple.

Now solve this pde ... just integrate twice

Integrate wrt. ξ $\quad u_\eta = \phi(\eta)$ any function of η would work

Integrate wrt. η $\quad u = \Phi(\eta) + \psi(\xi)$

antiderivative — and function of ξ

This tells us that
The solution is the sum of some function of η plus some function of ξ.

$$u(\xi, \eta) = \phi(\eta) + \psi(\xi)$$

Now Lets go back to the original coordinates.

And: $\psi(x) = f(x) - \phi(x)$

so $$\psi(x) = \frac{f(x)}{2} + \frac{1}{2c}\int_{x_0}^{x} g(\xi)\,d\xi + K/2c.$$

FINALLY WE CAN PUT OUR SOLN BACK TOGETHER!

$$u(x,t) = \phi(x-ct) + \psi(x+ct)$$

$$= \tfrac{1}{2}f(x-ct) - \frac{1}{2c}\int_{x_0}^{x-ct} g(\xi)\,d\xi - \underline{K/2c} + \tfrac{1}{2}f(x+ct) + \frac{1}{2c}\int_{x_0}^{x+ct} g(\xi)\,d\xi + \underline{K/2c}$$

$$= \tfrac{1}{2}\left[f(x-ct) + f(x+ct)\right] + \frac{1}{2c}\int_{x-ct}^{x_0} g(\xi)\,d\xi + \frac{1}{2c}\int_{x_0}^{x+ct} g(\xi)\,d\xi$$

$$\boxed{u(x,t) = \tfrac{1}{2}\left[f(x-ct) + f(x+ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(\xi)\,d\xi.}$$

This is the D'Alembert Soln to the wave equation.

- How do we use this ... PLUG IN!

Ex Motion of an initial sine wave.

$$\left.\begin{aligned} u_{tt} &= c^2 u_{xx} \\ u(x,0) &= \sin(x) \\ u_t(x,0) &= 0 \end{aligned}\right\}$$ vibrating infinite string with initial sine wave.

here $f(x) = \sin(x)$ and $g(x) = 0$

so
$$u(x,t) = \tfrac{1}{2}\left[\sin(x-ct) + \sin(x+ct)\right]$$

each piece of the wave goes in its own direction.

<u>EX</u> Initial velocity.

$$\left.\begin{array}{l} u_{tt} = c^2 u_{xx} \\ u(x,0) = 0 \\ u_t(x,0) = \sin(x) \end{array}\right\}$$ the string starts flat with some initial velocity.

$$u(x,t) = \frac{1}{2c}\int_{x-ct}^{x+ct} \sin(s)\,ds = \frac{1}{2c}\cos(s)\Big|_{x-ct}^{x+ct}$$

$$= \frac{1}{2c}\left[\cos(x+ct) - \cos(x-ct)\right]$$

<u>EX</u> Square wave

$$u_{tt} = c^2 u_{xx}$$

$$u(x,0) = \begin{cases} 1 & -1 < x < 1 \\ 0 & \text{otherwise} \end{cases} = f(x)$$

$$u_t(x,0) = 0$$

Partial Differential Equations

Professor Bieri
joanna_bieri@redlands.edu

Homework and Reading - Day 17:

Due Tuesday 11/12

Read and Take Notes

1. *Farlow* - Lesson 17
2. *Farlow* - *read ahead* - You can read ahead of the class if you want. The next chapters we will be covering are Lessons 16 and 19.

Homework

1. *Problems* - Do problems $1, 3, 4$ at the end of Chapter 17.
2. *Review* - Write a "cheat sheet" for the exam. I imagine this as being a flow chart for how to solve PDEs. For example, given a PDE:
    - How do you choose a solution method?
    - Once you pick the solution method how to you use it?
    - What are things you need to be careful of or that give you trouble?
3. *Areas to Study*
    - Classify a PDE: order, linear or not, homogeneous, etc...
    - Solving Boundary Value Problems for Eigenvalues and Eigenfunctions
    - Separation of Variables for the Heat Equation and Laplace's Equation
    - Eigenfunction Expansion
    - Transforming Equations:
        - Turning Nonhomogeneous BCs into Homogeneous ones
        - Turning Hard Problems into Easy Ones (Transform out heat loss and convection)
        - Using coordinate Transformations
    - Integral Transforms
        - Most important: Laplace and Fourier Transforms
        - Less important: Sine and Cosine Transforms
        - Most important: Transforming the PDE and solving the new problem.
        - Less important: Calculating the transform and solving those nasty integrals
4. *Not on Exam*
    - Deriving Heat Equation or Wave Equation
    - Duhamel's Principle
    - The Wave Equation