DAY 20.

Exam

Lesson 19 – So far we have solved the wave equation on the infinite domain $-\infty < x < \infty$.

Now we start discussing possible boundary conditions for the wave eqn.

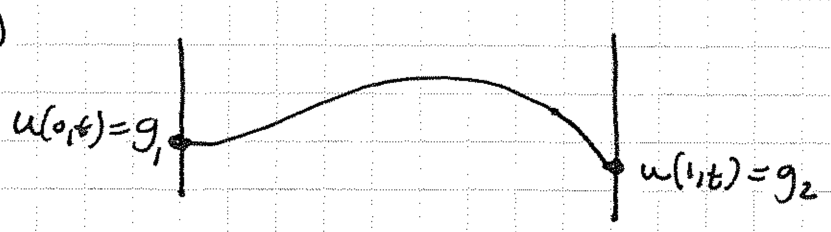
Lots of different physical systems are represented by the wave eqn:

1. Sound Waves
2. Electromagnetic waves
3. Vibrations in solids
4. Probability waves
5. Fluid Waves
6. Vibrating strings

When thinking about boundary conditions think physically about these types of problems.

Controlled End Points.

$$\left.\begin{aligned} &u_{tt} = c^2 u_{xx} \\ &u(0,t) = g_1(t) \\ &u(1,t) = g_2(t) \\ &u(x,0) = f(x) \\ &u_x(x,0) = g(x) \end{aligned}\right\} \text{ the location of the end points is specified.}$$



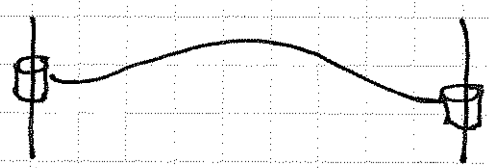
sometimes the goal is to find g_2 such that even though a string is shaking it eventually goes to zero.

Force on the Boundaries.

- Last time we showed that the vertical force due to tension was $T u_x$ so specifying force means specifying the first derivative.

$u_x(0,t) = 0$
$u_x(L,t) = 0$

No forcing or frictionless motion.



The string is held tight but any tension in y is lost to the movement of the ends.

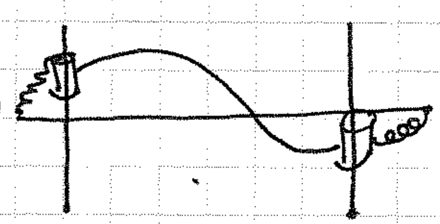
$u_x(0,t) = V(t)$

$\hookrightarrow$ forcing is applied at one ~~or~~ more boundaries.

FOR EXAMPLE: an electric field applied still allows movement but forcing is felt.

Elastic Attachment.

Displacement ~~$\propto$~~ tension



Displacement: $u(0,t)$ $\quad$ $u(L,t)$

Tension: $T u_x(0,t)$ $\quad$ $-T u_x(L,t)$

choose one side as upward. or one direction as positive x.

This means that $\quad$ $h u(0,t) = T u_x(0,t)$ $\quad$ h = spring constant.

Solving for the homogeneous form

$$u_x - \frac{h}{T} u(0,t) = 0$$

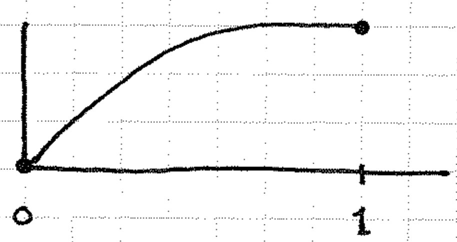
$$u_x(L,t) + \frac{h}{T} u(L,t) = 0$$

NOTE... we could imagine much more complicated cases based on forcing, material properties, and attachments.

Ex. $u_{tt} = c^2 u_{xx}$
$u(0,t) = 0$
$u(1,t) = 1$
$u(x,0) = \sin\left(\frac{\pi x}{2}\right)$
$u_t(x,0) = 0$

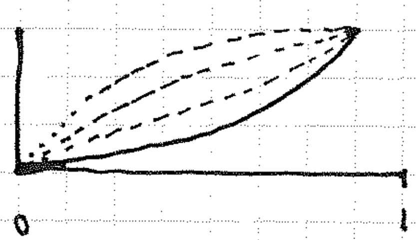
What should the solution look like?

Fixed ends



$t=0$

then the PDE takes over and the string will move based on curvature



$t = $ later

it will vibrate along the line $y=x$.

Now Lets jump right in and solve the problem of a finite vibrating string.

$u_{tt} = \alpha^2 u_{xx}$

$u(0,t) = 0 \longrightarrow$ fixed homogeneous boundaries.

$u(L,t) = 0$

$\left.\begin{aligned} u(x,0) &= f(x) \\ u_t(x,0) &= g(x) \end{aligned}\right\}$ genearne initial position and velocity

- similar to the heat equation ... as long as the spatial components are homogeneous we can separate variables.

Assume $u(x,t) = T(t)\,X(t)$ — <u>You</u> plug in and separate variables.

$T''X = \alpha^2 X'' T$ or $\dfrac{T''}{\alpha^2 T} = \dfrac{X''}{X} = \lambda$ or $\begin{aligned} &T'' - \alpha^2 \lambda T \\ &X'' - \lambda X = 0 \end{aligned}$

$\left.\begin{aligned} &X'' - \lambda X = 0 \\ &X(0) = 0 \\ &X(L) = 0 \end{aligned}\right\rangle$ Solve the Eigenvalue problem.

Can't separate the non-homogeneous initial conditions

$T'' - \alpha^2 \lambda T = 0$

— now test all cases for λ !!

solutions could grow if we have resonance...

$\lambda > 0 \quad X(x) = Ae^{-\sqrt{\lambda}t} + Be^{\sqrt{\lambda}t}$

but applying bc... $\underset{\text{Trivial}}{X=0}$

$\lambda = 0 \quad X(x) = Ax + B$

but applying bc, $\underset{\text{trivial}}{X=0}$

$\lambda < 0 \quad X(x) = A\sin(\sqrt{\lambda}x) + B\cos(\sqrt{\lambda}x)$

$X(0) = B = 0$

$X(L) = A\sin(\sqrt{\lambda}L) = 0$

$\sqrt{\lambda}L = n\pi$

$\sqrt{\lambda} = \dfrac{n\pi}{L}$

$X = \sin\left(\dfrac{n\pi x}{L}\right)$ and $\lambda = -\dfrac{(n\pi)^2}{L^2} \quad n = 1,2,3\ldots$

NOW SOLVE

$T'' + \left(\dfrac{\alpha n\pi}{L}\right)^2 T = 0$

$T_n = A_n \sin\left(\dfrac{\alpha n\pi}{L}t\right) + B_n \cos\left(\dfrac{\alpha n\pi}{L}t\right)$

$$u(x,t) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{L}\right)\left[A_n \sin\left(\frac{\alpha n\pi}{L}t\right) + B_n \cos\left(\frac{\alpha n\pi}{L}t\right)\right]$$

Apply the Initial Conditions...

$$\boxed{\begin{aligned} u(x,0) &= \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{L}\right)\left[B_n\right] = f(x) \\ u_t(x,0) &= \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{L}\right)\left[\frac{\alpha n\pi}{L} A_n\right] = g(x) \end{aligned}}$$

now use orthogonality...

$$\int_0^L \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right) dx = \int_0^L f(x) \sin\left(\frac{m\pi x}{L}\right) dx$$

$m=n$

$$\frac{LB_n}{2} = \int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx \quad \text{or} \quad B_n = \frac{2}{L}\int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right)dx$$

$$\int_0^L \sum_{n=1}^{\infty} A_n \frac{\alpha n\pi}{L} \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right)dx = \int_0^L g(x) \sin\left(\frac{m\pi x}{L}\right)dx$$

$m=n$ $$\frac{\alpha n\pi}{2} A_n = \int_0^L g(x) \sin\left(\frac{n\pi x}{L}\right) dx \quad \text{or} \quad A_n = \frac{2}{\alpha n\pi}\int_0^L g(x) \sin\left(\frac{n\pi x}{L}\right)dx$$

So given our initial conditions we can evaluate these integrals and reach our soln.

Solution to the VIBRATING STRING PROBLEM...

$$u(x,t) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{L}\right)\left[A_n \sin\left(\frac{\alpha n\pi}{L}t\right) + B_n \cos\left(\frac{\alpha n\pi}{L}t\right)\right]$$

use this for problems 1, 2, and 4

$$A_n = \frac{2}{\alpha n\pi}\int_0^L g(x) \sin\left(\frac{n\pi x}{L}\right) dx$$

$$B_n = \frac{2}{L}\int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx.$$

Partial Differential Equations

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Homework and Reading - Day 20:
Due Thursday 11/14

Read and Take Notes

1. *Farlow* - Lesson 20
2. *Farlow - read ahead* - You can read ahead of the class if you want. The next chapter we will be covering is Lesson 21 on the fourth order beam equation.

Homework

1. *Problems* - Do problems 1 and 2 at the end of Chapter 19.
2. *Problems* - Do problems $1, 2, 6$ and 7 at the end of Chapter 20.