To Solve the wave eqn in polar ... we will need BESSELS FUNCTIONS.

BESSELS FUNCTIONS – are solutions to Bessels Differential Equation

$$x^2y'' + xy' + (x^2-p^2)y = 0$$

where $p$ is a constant.

This ODE arises often when solving problems of $\nabla^2 u$ in polar coordinates.

To do this we use a special technique in odes

METHOD of FROBENIUS

OUTLINE SOLEV: $A(x)y'' + B(x)y' + C(x)y = 0$

by assuming $y = \sum_{m=0}^{\infty} c_m x^{m+r}$

* NOTE.. This is a special case of the POWER SERIES METHOD – When $A(x) \neq 0$ for any $x$ in the domain $y = \sum_{n=1}^{\infty} c_n x^n$ plug in find $c_n$.

For this to work we need to assume some things about $A(x)$, $B(x)$, $C(x)$.

$$\lim_{x\to x_0} (x-x_0)\frac{B(x)}{A(x)} \qquad \lim_{x\to x_0} (x-x_0)^2 \frac{C(x)}{A(x)}$$

both finite.

Bessel's Eqn $A(x) = x^2$, $B(x) = x$, $C(x) = (x^2-p^2)$

problem at $x=0$ $A(x)=0$ and we "loose" our $y''$.

method of frobenius will work if ...

$$\lim_{x\to 0} x\frac{x}{x^2} = \lim_{x\to 0} 1 = 1 \quad \text{Finite} \checkmark$$

$$\lim_{x\to 0} (x)^2\frac{(x^2-p^2)}{x^2} = \lim_{x\to 0} x^2-p^2 = -p^2 \quad \text{Finite} \checkmark$$

Plug $y = \sum_{m=0}^{\infty} C_m x^{m+r}$ into Bessels ODE

$$y' = \sum_{m=0}^{\infty} C_m (m+r) x^{m+r-1} \qquad y'' = \sum_{m=0}^{\infty} C_m (m+r)(m+r-1) x^{m+r-2}$$

- we assume here that y converges on the open interval I so that y is differentiable on I and the power series can be directly differentiated.

so our ODE becomes:

$$\sum_{m=0}^{\infty} C_m (m+r)(m+r-1) x^{m+r} + \sum_{m=0}^{\infty} C_m (m+r) x^{m+r} +$$

$$\sum_{m=0}^{\infty} C_m x^{m+r+2} - p^2 \sum_{m=0}^{\infty} C_m x^{m+r} = 0$$

shift the index $= \sum_{m=2}^{\infty} C_{m-2} \underline{\underline{x^{m+r}}}$ same form

this helps us gather terms! $x^r, x^{r+1}, x^{r+2}$, etc.

$$\sum_{m=0}^{\infty} C_m\left[(m+r)(m+r-1) + (m+r) - p^2\right] x^{m+r}$$

$$+ \sum_{m=2}^{\infty} C_{m-2} x^{m+r} = 0$$

now $x^r$ terms ar $m=0$, $x^{r+1}$ are $m=1$, etc and the coefficients of each of those must go to zero.

$m=0$ $\quad C_0\left[r(r-1) + r - p^2\right]x^r = C_0\left[r^2 - p^2\right]x^r = 0$ so $C_0 = 0$ or $r^2 - p^2 = 0$

$m=1$ $\quad C_1\left[(1+r)(r) + (1+r) - p^2\right]x^{1+r} = C_1\left[r^2 + 2r + 1 - p^2\right] = 0$

so $C_1 = 0$ or $r^2 + 2r + 1 - p^2 = 0$

$m \geq 2$ $\quad C_m\left[(m+r)\overset{(1+r)}{(m+r-1)} + (m+r) - p^2\right] + C_{m-2} = 0$

$m^2 + mr + mr + r^2 - m - r + m + r$

$$C_m\left[(m+r)^2 - p^2\right] + C_{m-2} = 0 \quad \text{or}$$

$$C_m = \frac{-C_{m-2}}{\left[(m+r)^2 - p^2\right]}$$

Ok First ... we don't want our leading term in the series to go to zero!

$C_0 \neq 0$ so this relates $r$ and $p$ by the characteristic equation $r^2 - p^2 = 0$

or $(r+p)(r-p) = 0$

or $r = \pm p$

Second if $(r^2 - p^2) = 0$ then

$C_1 = 0$ can't avoid this.

The rest are found through the recurrence relation

$$C_m = \frac{-C_{m-2}}{[(m+r)^2 - p^2]}$$

What do we have so far... two linearly independent solns of the form

$$y = \sum_{m=0}^{\infty} C_m x^{m+r}$$

for $r = \pm p$ and $C_1 = 0$

$$C_m = \frac{-C_{m-2}}{[(m+r)^2 - p^2]}$$

note $C_0$ still undefined and $C_m$ for $m$ = even only.

CASE: $r = p$

then $C_0 = C_0$

$$C_m = \frac{-C_{m-2}}{[m^2 + 2pm]} = \frac{-C_{m-2}}{m[m+2p]}$$

$$C_2 = \frac{-C_0}{2(2+2p)} = \frac{-C_0}{2^2(1+p)}$$

$$C_4 = \frac{-C_2}{4(4+2p)} = \frac{-C_2}{2^2 \cdot 2(2+p)} = \frac{C_0}{2^4 \cdot 2(2+p)(1+p)}$$

$$C_6 = \frac{-C_4}{6[6+2p]} = \frac{-C_4}{2 \cdot 6(3+p)} = \frac{C_0}{2^6 \cdot 2 \cdot 3(3+p)(2+p)(1+p}$$

look for the pattern:

$$C_{2m} = \frac{(-1)^m C_0}{2^{2m} m!(p+1)(p+2)\cdots(p+m)}$$

so $$y_1(x) = C_0 \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m+p}}{2^{2m} m!(p+1)(p+2)\cdots(p+m)}$$ This is <u>one</u> linearly independent soln.

CASE: $r=-p$ then

$$y_2(x) = b_0 \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m-p}}{2^{2m} m!(-p+1)(-p+2)\cdots(-p+m)}$$

these converge for all $x>0$ <u>and</u> these are Bessels Functions.

We often rewrite these using the gamma function. and give them their own variable:

$$J_p(x) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\,\Gamma(p+m+1)} \left(\frac{x}{2}\right)^{2m+p}$$

$$J_{-p}(x) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\,\Gamma(-p+m+1)} \left(\frac{x}{2}\right)^{2m-p}$$

BESSELS FUNCTIONS OF THE FIRST KIND

so our soln should be written

$$y = C_1 J_p(x) + C_2 J_{-p}(x)$$

NOW... these solutions are only linearly independent when $p \neq$ integer.

Ex. $J_{1/2}(x) = \sqrt{\frac{2}{\pi x}}\sin(x)$ $\quad J_{-1/2} = \sqrt{\frac{2}{\pi x}}\cos(x)$

BUT if $p$ is an integer then $J_{-p}(x) = (-1)^p J_p(x)$ and we actually need to do more work to find a linearly independent soln!

If we did this it would be complicated but we could show that.

$$Y_p(x) = \frac{J_p(x)\cos(\pi p) - J_{-p}(x)}{\sin(\pi p)}$$

BESSELS FUNCTIONS OF THE SECOND KIND

GRAPHS of $J_p$ and $J_{-p}$ and $Y_p$ on computer... .m files.

NOTE... these are oscillating solutions... even though they are expressed as sums!