PDEs — DAY 21

Fourth order PDEs.

$$u_{tt} = -\alpha^2 u_{xxxx}$$ Vibrating beam problem.
The beam offers resistance to bending $\rightarrow$ fourth order.

$$\alpha^2 = K/\rho = \frac{\text{rigidity}}{\text{linear density}}.$$

Assume the beam is simply fastened — held tight but the slope can change.

$$\text{BC's} \longrightarrow \left.\begin{aligned} u(0,t) &= 0 \\ u(1,t) &= 0 \end{aligned}\right\} \text{held tight.}$$

problem... we need two more BC's.
we require that

$$\left.\begin{aligned} u_{xx}(0,t) &= 0 \\ u_{xx}(1,t) &= 0 \end{aligned}\right\}$$ The curvature is zero — Why. Zero bending Moment.

Visually — 

when the beam moves the end will wiggle and try to remain straight.

$$\left.\begin{aligned} &u_t = -u_{xxxx} && \text{let } \alpha = 1 \\ &u(0,t) = 0 \\ &u(1,t) = 0 && \text{Four} \\ &u_{xx}(0,t) = 0 && \text{BCs} \\ &u_{xx}(1,t) = 0 \\ \\ &u(x,0) = f(x) && \text{Two} \\ &u_t(x,0) = g(x) && \text{ICs} \end{aligned}\right\}$$ Solve using separation of variables!

let $u(x,t) = X(x)T(t)$
plug in

$$T'X = -X^{(4)}T$$
$$-\frac{T'}{T} = \frac{X^{(4)}}{X} = K$$

$$T'' + KT = 0 \quad \text{and} \quad \left.\begin{aligned} X^{(4)} - KX = 0 \\ 4 \text{ BC's.} \end{aligned}\right\}$$ eigenvalue problem!

NEW PROBLEM

$$\left.\begin{aligned} &X^{(4)} - kX = 0 \\ &X(0) = 0 \\ &X(1) = 0 \\ &X''(0) = 0 \\ &X''(1) = 0 \end{aligned}\right\}$$

$r^4 - K = 0$

- we expect vibrating, sinusoidal solns ... esp because of the first two BC's.

let $K = w^2$ to make the calcs easier ... assume $K \geq 0$

$$r^4 - w^2 = (r^2 + w)(r^2 - w) = 0$$

$$r^2 = -w \quad \text{or} \quad r = \pm i\sqrt{w}$$

$$r^2 = w \qquad\qquad r = \pm\sqrt{w}$$

$w = 0$ Four repeated roots: $X(x) = Ax^3 + Bx^2 + Cx + D$

Easier to apply conditions at zero first

$X(0) = D = 0$

$X''(0) = 2B = 0$

$X(x) = Ax^3 + Cx$

$X(1) = A + C = 0$ or $A = -C$ $\quad X(x) = C(x - x^3)$

$X''(1) = -6C = 0 \quad C = 0$ $\quad$ TRIVIAL!

$w \neq 0$ then:

$$X(x) = A\sin(\sqrt{w}x) + B\cos(\sqrt{w}x) + \underbrace{Ce^{\sqrt{w}x} + De^{-\sqrt{w}x}}$$

we can also write this as

$C\sinh(\sqrt{w}x) + D\cosh(\sqrt{w}x)$.

Apply BC's.

$X(0) = B + D = 0$

$X''(x) = -Aw\sin(\sqrt{w}x) - Bw\cos(\sqrt{w}x) + Cw\sinh(\sqrt{w}x) + wD\cosh(\sqrt{w}x)$

$X''(0) = -B + D = 0$

This means that $B = D = 0$!

$$X(x) = A\sin(\sqrt{w}x) + C\sinh(\sqrt{w}x).$$

$X(1) = A\sin(\sqrt{w}) + C\sinh(\sqrt{w}) = 0$

$X''(1) = -A\sin(\sqrt{w}) + C\sinh(\sqrt{w}) = 0$

$A\sin(\sqrt{w}) = 0$ and $C\sinh(\sqrt{w}) = 0$

$\sqrt{w} = n\pi$ $\qquad C = 0$

or $w = (n\pi)^2$

or $K = (n\pi)^4$

Notes on $\sinh(x)$ and $\cosh(x)$...

ASIDE

$f(x) = \sinh(x) = \frac{1}{2}(e^{x} - e^{-x})$

$f(0) = 0$

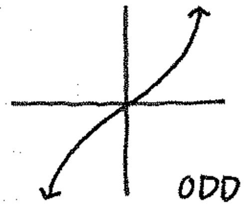
$f'(x) = \cosh(x)$

no negatives out of derivatives.

$g(x) = \cosh(x) = \frac{1}{2}(e^{x} + e^{-x})$

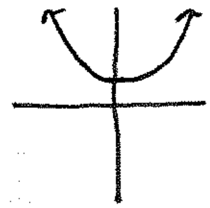
$g(0) = 1$

$g'(x) = \sinh(x)$



ODD

NOT PERIODIC



EVEN

$\sinh(x) = sh(x)$

$\cosh(x) = ch(x)$.

Soln in X: $X(x) = A\sin(n\pi x)$ and $K = (n\pi)^4$

NOW SOLVE: $T'' + (n\pi)^4 T = 0$

$$T(t) = A\sin((n\pi)^2 t) + B\cos((n\pi)^2 t)$$

And $$u(x,t) = \sum_{n=1}^{\infty} \left(A_n \sin[n^2\pi^2 t] + B_n \cos[n^2\pi^2 t]\right)\sin(n\pi x)$$

notice the higher frequency of vibration here!!

Now use ICs to ~~fix~~ solve for A_n and B_n.

$$u(x,0) = \sum_{n=1}^{\infty} B_n \sin(n\pi x) = f(x)$$

$$u_t(x,0) = \sum_{n=1}^{\infty} n^2\pi^2 A_n \sin(n\pi x) = g(x)$$

we have solved this a million times.

$$B_n = 2\int_0^1 f(x)\sin(n\pi x)\,dx$$

$$A_n = \frac{2}{n^2\pi^2}\int_0^1 g(x)\sin(n\pi x)\,dx$$

EX What if $u(x,0) = \sin(\pi x) + \frac{1}{2}\sin(3\pi x)$
$u_t(x,0) = 0$

By observation $B_1 = 1 \quad B_2 = 0 \quad B_3 = 1/2 \quad B_4 = 0 \ldots$
$A_n = 0$

so our soln is...

$$u(x,t) = \cos(\pi^2 t)\sin(\pi x) + \frac{1}{2}\cos(9\pi^2 t)\sin(3\pi x)$$

compare This to the vibrating string from last time...

$$u(x,t) = \cos(\pi t)\sin(\pi x) + \frac{1}{2}\cos(3\pi t)\sin(3\pi x)$$

MATLAB.

Think about this physically ... it makes sense.

Try another one... Left end rigidly fastened $u(0,t)=0$, $u_x(0,t)=0$

Right end simply fastened $u(1,t)=0$, $u_{xx}(1,t)=0$

#4 from the Homework.

Now we are solving:

$$\left.\begin{aligned} &X^{(4)} - kX = 0 \\ &X(0) = 0 \\ &X'(0) = 0 \\ &X(1) = 0 \\ &X''(1) = 0 \end{aligned}\right\}$$

From the last example $k = w^2$

$$X(x) = A\sin(\sqrt{w}x) + B\cos(\sqrt{w}x) + C\sinh(\sqrt{w}x) + D\cosh(\sqrt{w}x)$$

OR

$$X(x) = Ax^3 + Bx^2 + Cx + D.$$

$w = 0$

$$\left.\begin{aligned} &X(0) = D = 0 \qquad X(x) = Ax^3 + Bx^2 \\ &X'(0) = C = 0 \\ &X(1) = A + B = 0 \quad A = -B \quad X(x) = B(x^2 - x^3) \\ &X''(1) = B(2-6) = 0 \qquad B = 0 \end{aligned}\right\}$$ still trivial.

$w \neq 0$

$$\left.\begin{aligned} &X(0) = B + D = 0 \quad B = -D \\ &X'(0) = A\sqrt{w} + C\sqrt{w} = 0 \quad A = -C \end{aligned}\right\}$$

$$X(x) = A(\sin(\sqrt{w}x) - \sinh(\sqrt{w}x)) + B(\cos(\sqrt{w}x) - \cosh(\sqrt{w}x))$$

$$X(1) = A\left(\sin(\sqrt{\omega}) - \sinh(\sqrt{\omega})\right) + B\left(\cos(\sqrt{\omega}) - \cosh(\sqrt{\omega})\right) = 0$$
$$X''(1) = -A\omega\left(\sin(\sqrt{\omega}) + \sinh(\sqrt{\omega})\right) - \omega B\left(\cos(\sqrt{\omega}) + \cosh(\sqrt{\omega})\right) = 0$$

$$AS + BC = 0 \qquad A = -\frac{BC}{S}$$
$$-AS^* - BC^* = 0$$

$$\frac{BC}{S}S^* - BC^* = 0$$

$$B\left(\frac{CS^*}{S} - C^*\right) = 0$$

$B = 0$ would mean trivial.

$$\frac{CS^*}{S} - C^* = 0 \Rightarrow \frac{\left(\cos(\sqrt{\omega}) - \cosh(\sqrt{\omega})\right)}{\left(\sin(\sqrt{\omega}) - \sinh(\sqrt{\omega})\right)}\left(\sin(\sqrt{\omega}) + \sinh(\sqrt{\omega})\right) - \left(\cos(\sqrt{\omega}) + \cosh(\sqrt{\omega})\right) = 0$$

$$\frac{\cos\sqrt{\omega} - \operatorname{ch}\sqrt{\omega}}{\cos\sqrt{\omega} + \operatorname{ch}\sqrt{\omega}} \cdot \frac{\sin\sqrt{\omega} + \operatorname{sh}\sqrt{\omega}}{\sin\sqrt{\omega} - \operatorname{sh}\sqrt{\omega}} = 1$$

$$\frac{\cos\sqrt{\omega} - \operatorname{ch}\sqrt{\omega}}{\cos\sqrt{\omega} + \operatorname{ch}\sqrt{\omega}} = \frac{\sin\sqrt{\omega} - \operatorname{sh}\sqrt{\omega}}{\sin\sqrt{\omega} + \operatorname{sh}\sqrt{\omega}}$$

$$\cancel{\cos\sqrt{\omega} - \operatorname{ch}\sqrt{\omega} = \sin\sqrt{\omega} - s}$$

$$\left(\cos\sqrt{\omega} - \operatorname{ch}\sqrt{\omega}\right)\left(\sin\sqrt{\omega} + \operatorname{sh}\sqrt{\omega}\right) = \left(\sin\sqrt{\omega} - \operatorname{sh}\sqrt{\omega}\right)\left(\cos\sqrt{\omega} + \operatorname{ch}\sqrt{\omega}\right)$$

$$\cancel{\cos\sqrt{\omega}\sin\sqrt{\omega}} - \operatorname{ch}\sqrt{\omega}\sin\sqrt{\omega} + \operatorname{sh}\sqrt{\omega}\cos\sqrt{\omega} - \cancel{\operatorname{ch}\sqrt{\omega}\operatorname{sh}\sqrt{\omega}}$$
$$= \cancel{\sin\sqrt{\omega}\cos\sqrt{\omega}} - \operatorname{sh}\sqrt{\omega}\cos\sqrt{\omega} + \sin\sqrt{\omega}\operatorname{ch}\sqrt{\omega} - \cancel{\operatorname{sh}\sqrt{\omega}\operatorname{ch}\sqrt{\omega}}$$

$$2\operatorname{ch}\sqrt{\omega}\sin\sqrt{\omega} - 2\operatorname{sh}\sqrt{\omega}\cos\sqrt{\omega} = 0$$

$$\operatorname{ch}\sqrt{\omega}\sin\sqrt{\omega} = \operatorname{sh}\sqrt{\omega}\cos\sqrt{\omega}$$

natural frequencies satisfy this.

The Lagrangian:

Assume no External Forces.

$$\mathcal{L} := \frac{1}{2}\mu\left(\frac{\partial w}{\partial t}\right)^2 - \frac{1}{2}EI\left(\frac{\partial^2 w}{\partial x^2}\right)^2$$

Kinetic Energy

potential Energy from Internal forces

relationship between curvature and bending moment.

$$= \frac{1}{2}\mu\,\dot{w}^2 - \frac{EI}{2}w_{xx}^2$$

Euler-Lagrange:

$$\frac{\partial \mathcal{L}}{\partial w} - \frac{\partial}{\partial t}\left(\frac{\partial \mathcal{L}}{\partial \dot{w}}\right) + \frac{\partial^2}{\partial x^2}\left(\frac{\partial \mathcal{L}}{\partial w_{xx}}\right) = 0$$

$$0 \qquad \mu\dot{w} \qquad -EI\,w_{xx}$$

$$-\frac{\partial}{\partial t}\mu\dot{w} - \frac{\partial^2}{\partial x^2}(+EI\,w_{xx}) = 0$$

$$-\mu\ddot{w} - EI\,w_{xxxx} = 0$$

assuming a homogeneous beam.

<u>so</u> $w_{tt} = -\alpha^2 w_{xxxx}$

where $\alpha^2 = \frac{EI}{\mu}$

So the fourth derivative arises because the potential energy in the beam is dependent on curvature.

<u>NOTE</u> - Euler Lagrange eqn seeks to find the minimum path from an energy standpoint. In general.

$$\frac{\partial \mathcal{L}}{\partial f} - \frac{d}{dx}\frac{\partial \mathcal{L}}{\partial f'} + \frac{d^2}{dx^2}\left(\frac{\partial \mathcal{L}}{\partial f''}\right) - \cdots + (-1)^n\frac{d^n}{dx^n}\left(\frac{\partial \mathcal{L}}{\partial f^{(n)}}\right) = 0$$

where $\mathcal{L}(x, f, f', f'', \ldots, f^{(n)})$

in the most simple case → this can reduce to the statement that a max or min happens when the derivative is zero.