Classification and Canonical Forms.

Goal – Classify second order PDEs.

General Form $Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu = G$

Hyperbolic: $B^2 - 4AC > 0$
Parabolic: $B^2 - 4AC = 0$
Elliptic: $B^2 - 4AC < 0$
} solution methods and possible canonical forms depend on Classification.

Ex $u_t = u_{xx} \longrightarrow u_{xx} - u_t = 0$ here $y = t$

$A = 1$ $E = -1$ all others zero
$0^2 - 4\,1(0) = 0$ PARABOLIC!

Ex $u_{tt} = u_{xx} \longrightarrow u_{xx} - u_{tt} = 0$

$A = 1$ $C = -1$ all others zero
$0^2 - 4(1)(-1) = 4 > 0$ HYPERBOLIC!

Ex $u_{xx} + u_{yy} = 0$
$A = 1$ $B = 0$ $C = 1$ all others zero
$0^2 - 4(1)(1) = -4 < 0$ ELLIPTIC!

Ex $x u_{xx} + u_{yy} = \sin(x)$ variable coefficients

$A = x$ $C = 1$ all others zero
$0^2 - 4(x)(1) = -4x$

$x < 0$ Hyperbolic
$x = 0$ Parabolic
$x > 0$ Elliptic

Each Equation Type has a specific CANONICAL FORM.

HYPERBOLIC $\longrightarrow$ $u_{\xi\xi} - u_{\eta\eta} = \psi(\xi, \eta, u, u_\xi, u_\eta)$
$u_{\xi\eta} = \phi(\xi, \eta, u, u_\xi, u_\eta)$

PARABOLIC $\longrightarrow$ $u_{\eta\eta} = \phi(\xi, \eta, u, u_\xi, u_\eta)$

ELLIPTIC $\longrightarrow$ $u_{\xi\xi} + u_{\eta\eta} = \phi(\xi, \eta, u, u_\xi, u_\eta)$

Today – tricks for simplifying more complicated PDEs.

Nondimensionalization – often it's easier to deal with problems where the dimensions have been taken out, or normalized.

Transform seemingly different problems into the same form.

Ex. Diffusion Problem

$$\left.\begin{aligned} &u_t = \alpha^2 u_{xx} \\ &u(0,t) = T_1 \\ &u(L,t) = T_2 \\ &u(x,0) = \sin\left(\tfrac{\pi x}{L}\right) \end{aligned}\right\}$$

Want to change the problem
1. No physical parameters
2. BCs and ICs are simpler

Do this through a change of variables:

$$u(x,t) \longrightarrow U(\xi,\tau).$$

First define a new Temperature.

$$U(x,t) = \frac{u(x,t) - T_1}{T_2 - T_1}$$

The new "temperature" has no units.

Use reference temperatures from the boundaries.

Transform Each Piece.

$$u(x,t) = (T_2 - T_1)U(x,t) + T_1$$

$$u_t = (T_2 - T_1)U_t \quad u_{xx} = (T_2 - T_1)U_{xx}$$

$$u(0,t) = (T_2 - T_1)U(0,t) + T_1 = T_1 \quad U(0,t) = 0$$

$$u(1,t) = (T_2 - T_1)U(1,t) + T_1 = T_2 \quad U(1,t) = 1$$

$$u(x,0) = (T_2 - T_1)U(x,0) + T_1 = \sin\left(\tfrac{\pi x}{L}\right) \quad U(x,0) = \frac{\sin(\pi x/L) - T_1}{T_2 - T_1}$$

NEW PDE

$$\left.\begin{aligned} &U_t = \alpha^2 U_{xx} \quad 0<x<L \\ &U(0,t)=0 \\ &U(1,t)=1 \\ &U(x,0) = \frac{\sin(\pi x/L) - T_1}{T_2 - T_1} \end{aligned}\right\}$$

NOTE... nondims are not unique. We could have picked $U(x,t) = \frac{u(x,t)}{T_2}$

Transform space – choose a "length" variable with no units.

let $\xi = x/L$ * usually wrt the lenght of the domain of the diffusion length.

$$U_x = U_\xi \frac{d\xi}{dx} = \frac{U_\xi}{L} \quad \text{and} \quad U_{xx} = U_{\xi\xi}\left(\frac{d\xi}{dL}\right)^2 = \frac{U_{\xi\xi}}{L^2}$$

NEW PDE

$$\left.\begin{aligned} &U_t = \left(\frac{\alpha^2}{L^2}\right) U_{\xi\xi} \quad 0<\xi<1 \\ &U(0,t)=0 \\ &U(1,t)=1 \\ &U(\xi,0) = \frac{\sin(\pi\xi) - T_1}{T_2 - T_1} \end{aligned}\right\}$$

the domain is now between zero and one.

Transform time – choose a "time" variable that simplifies everything else.

let $\tau = ct$ plug in and solve for c.

$$U_t = U_\tau \frac{d\tau}{dt} = c\,U_\tau$$

so $c\,U_\tau = \frac{\alpha^2}{L^2} U_{\xi\xi}$ and $U_\tau = \frac{1}{c}\frac{\alpha^2}{L^2} U_{\xi\xi}$

so $c = \alpha^2/L^2$ making $\tau = \frac{\alpha^2}{L^2} t$

New PDE

$$\left.\begin{aligned} &U_\tau = U_{\xi\xi} \quad 0<\xi<1 \\ &U(0,\tau)=0 \\ &U(1,\tau)=1 \\ &U(\xi,0) = \phi(\xi) \end{aligned}\right\}$$

much simpler and more compact... probably a problem we have solved before!

– After finding the answer we transform back.

You Try...

$$\left.\begin{aligned} &u_{tt} = \alpha^2 u_{xx} \\ &u(0,t)=0 \\ &u(L,t)=0 \\ &u(x,0) = \sin(\pi x/L) + \tfrac{1}{2}\sin(3\pi x/L) \\ &u_t(x,0) = 0 \end{aligned}\right\}$$

Note transform only

$x \longrightarrow \xi$
$t \longrightarrow \tau$

because the BC's are already simple.

let $\xi = x/L$

Then $\frac{u_{\xi\xi}}{L^2} = u_{xx}$

Then solve by recognizing the problem from Lesson 20 prob 1

NEW PDE

$$\left.\begin{aligned} &u_{tt} = \alpha^2/L^2\, u_{\xi\xi} \\ &u(0,t)=0 \\ &u(1,t)=0 \\ &u(\xi,0) = \sin(\pi\xi) + \tfrac{1}{2}\sin(3\pi\xi) \\ &u_t(\xi,0) = 0 \end{aligned}\right\}$$

then let $\tau = ct$ and $u_{tt} = c^2 u_{\tau\tau}$

$$c^2 u_{\tau\tau} = \alpha^2/L^2\, u_{\xi\xi} \longrightarrow u_{\tau\tau} = \frac{1}{c^2}\frac{\alpha^2}{L^2} u_{\xi\xi}$$

This means that $c = \alpha/L$
$\tau = \alpha/L\, t$

NEW PDE.

$$\left.\begin{aligned} &u_{\tau\tau} = u_{\xi\xi} \\ &u(0,\tau)=0 \\ &u(1,\tau)=0 \\ &u(\xi,0) = \sin(\pi\xi) + \tfrac{1}{2}\sin(3\pi\xi) \\ &u_t(\xi,0) = 0 \end{aligned}\right\}$$

we solved this... $u(\xi,\tau) = \cos(\pi\tau)\sin(\pi\xi) + \tfrac{1}{2}\cos(3\pi\tau)\sin(3\pi\xi)$

giving the full soln:

$$u(x,t) = \cos\left(\frac{\pi\alpha t}{L}\right)\sin\left(\frac{\pi x}{L}\right) + \tfrac{1}{2}\cos\left(\frac{3\pi\alpha t}{L}\right)\sin\left(\frac{3\pi x}{L}\right)$$

Recall in the D'Alembert Soln.

$u_{tt} = c^2 u_{xx}$ we introduced $\xi = x + ct$, $\eta = x - ct$ $\longrightarrow$ $u_{\xi\eta} = 0$

we were switching to canonical form!!

* Doing this either greatly simplifies the PDE or sets us up for straight forward numerical analysis !!

<u>Canonical Form</u> – HYPERBOLIC ... want $u_{\xi\eta} = \phi(\xi, \eta, u, u_\xi, u_\eta)$

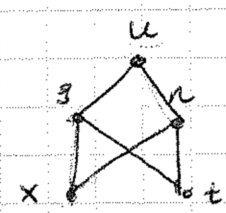
1. Define a coordinate transformation

$\xi = \xi(x, t)$ $\quad$ $\eta = \eta(x, t)$ $\longrightarrow$ goal of the method is to find these !!

Consider the general Form

$$A u_{xx} + B u_{xt} + C u_{tt} + D u_x + E u_t + F u = G$$

and sub in our new coordinates:



$u = u$
$u_x = u_\xi \xi_x + u_\eta \eta_x$
$u_t = u_\xi \xi_t + u_\eta \eta_t$

now the second derivatives.

$$u_{xx} = \frac{\partial}{\partial x}\left[u_x\right] = \frac{\partial}{\partial x}\left[u_\xi \xi_x + u_\eta \eta_x\right]$$

$$= u_{\xi\xi}(\xi_x)^2 + u_\xi \xi_{xx} + u_{\xi\eta}\xi_x \eta_x + u_{\eta\xi}\eta_x \xi_x + u_{\eta\eta}(\eta_x)^2 + u_\eta \eta_{xx}$$

$$= u_{\xi\xi}(\xi_x)^2 + 2u_{\xi\eta}\xi_x\eta_x + u_{\eta\eta}(\eta_x)^2 + u_\xi \xi_{xx} + u_\eta \eta_{xx}.$$

similarly ... on page 177 !!

$$u_{xt} = u_{\xi\xi}\xi_x\xi_t + u_{\xi\eta}(\xi_x\eta_t + \xi_t\eta_x) + u_{\eta\eta}\eta_x\eta_t + u_\xi \xi_{xt} + u_\eta \eta_{xt}$$
$$u_{tt} = u_{\xi\xi}\xi_t^2 + 2u_{\xi\eta}\xi_t\eta_t + u_{\eta\eta}\eta_t^2 + u_\xi \xi_{tt} + u_\eta \eta_{tt}$$

We substitute these into the general form to get:

$$\bar{A}u_{\xi\xi} + \bar{B}u_{\xi\eta} + \bar{C}u_{\eta\eta} + \bar{D}u_\xi + \bar{E}u_\eta + \bar{F}u = \bar{G}$$

where
$$\bar{A} = A\xi_x^2 + B\xi_x\xi_t + C\xi_t^2$$
$$\bar{B} = 2A\xi_x\eta_x + B(\xi_x\eta_t + \xi_t\eta_x) + 2C\xi_t\eta_t$$
$$\bar{C} = A\eta_x^2 + B\eta_x\eta_t + C\eta_t$$
$$\bar{D} = A\xi_{xx} + B\xi_{xt} + C\xi_{tt} + D\xi_x + E\xi_t$$
$$\vdots$$

again in The Book.

But NOW WE SET $\bar{A} = 0$ and $\bar{C} = 0$ to force the canonical Form.

This leads to

$$A\left[\xi_x/\xi_t\right]^2 + B\left[\xi_x/\xi_t\right] + C = 0$$
$$A\left[\eta_x/\eta_t\right]^2 + B\left[\eta_x/\eta_t\right] + C = 0$$

quadratic equations for ξ_x/ξ_t and η_x/η_t

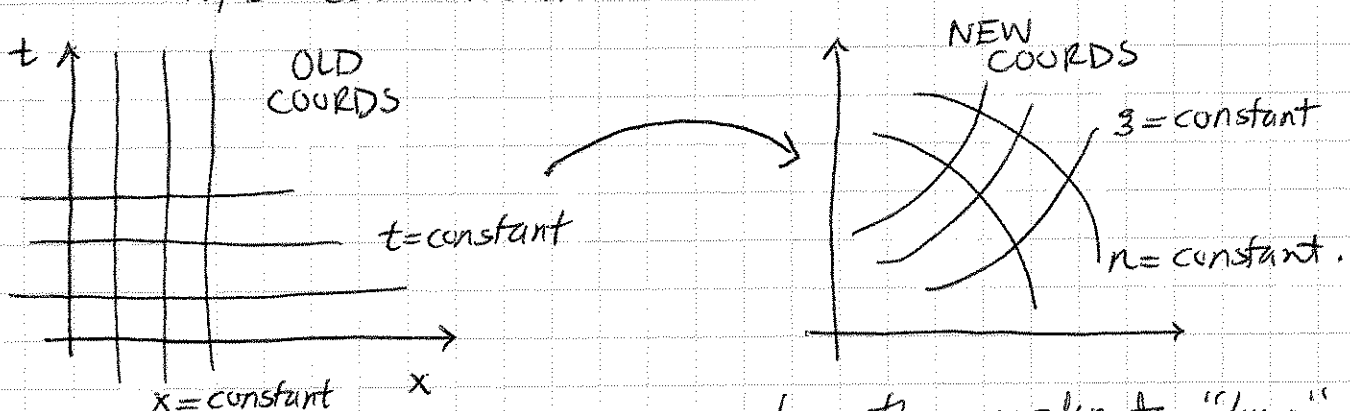
where $\left[\xi_x / \xi_t\right] = \dfrac{-B + \sqrt{B^2 - 4AC}}{2A}$

$$\left[\eta_x / \eta_t\right] = \frac{-B - \sqrt{B^2 - 4AC}}{2A}$$

note we just need one soln for each that will work and since we want different coordinates we choose one positive and one negative.

This is also where $B^2 - 4AC$ definitions come from. the idea of the characteristics.

NOW HOW DO WE RELATE THIS TO OUR ORIGINAL x, t COORDINATES.



along the coordinate "lines"

$\xi = \text{constant}$ $\quad\quad$ $\eta = \text{constant}$

$d\xi = \xi_x\,dx + \xi_t\,dt = 0 \quad\quad d\eta = \eta_x\,dx + \eta_t\,dt = 0$

OR

$$\frac{dt}{dx} = -\left[\xi_x / \xi_t\right] \quad\quad \frac{dt}{dx} = -\left[\eta_x / \eta_t\right]$$

we can use these to solve for ξ and η in terms of x and t.

Ex $u_{xx} - 4u_{tt} + u_x = 0$

$\left[\xi_x / \xi_t\right] = 2$ and $\left[\eta_x / \eta_t\right] = 2$

$\dfrac{dt}{dx} = -2 \quad\quad \dfrac{dt}{dx} = 2$

integrate

$t = -2x + C_1 \quad\quad t = 2x + C_2$

OR

$t + 2x = C_1 \quad\quad t - 2x = C_2$

$\xi = t + 2x = C_1 \quad\quad \eta = t - 2x = C_2$

So we can transform to canonical form using

$\xi = t + 2x$ and $\eta = t - 2x$!!