PDE — DAY 23

Lots of people still struggling with chain rule and change of variables... go slow! It is easy to drop terms!

IN DETAIL: $u(x,y) \longrightarrow u(\xi,\eta)$

but really $u(\xi,\eta) = u(\xi(x,y),\eta(x,y))$
hidden x and y's.

$$\frac{\partial}{\partial x} u(\xi,\eta) = \frac{\partial}{\partial x} u(\xi(x,y),\eta(x,y)) = \frac{\partial}{\partial \xi}[u]\frac{\partial \xi}{\partial x} + \frac{\partial}{\partial \eta}[u]\frac{\partial \eta}{\partial x}$$

to reach all the x's need chain rule!

General rule here $\quad \frac{\partial}{\partial x}[\ ] = \frac{\partial}{\partial \xi}[\ ]\xi_x + \frac{\partial}{\partial \eta}[\ ]\eta_x$

or

second derivatives. $\quad \frac{\partial}{\partial y}[\ ] = \frac{\partial}{\partial \xi}[\ ]\xi_y + \frac{\partial}{\partial \eta}[\ ]\eta_y \; (*)$

Try the mixed partial.

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x} = \frac{\partial}{\partial y}\left[\frac{\partial u}{\partial x}\right] = \frac{\partial}{\partial y}\left[\frac{\partial}{\partial x} u(\xi,\eta)\right]$$

we found this above. because we are changing vars.

$$\frac{\partial^2 u}{\partial y \partial x} = \frac{\partial}{\partial y}\left[\frac{\partial u}{\partial \xi}\xi_x + \frac{\partial u}{\partial \eta}\eta_x\right] = \frac{\partial}{\partial y}[u_\xi]\xi_x + u_\xi\frac{\partial}{\partial y}[\xi_x]$$

use the product rule.

$$+ \frac{\partial}{\partial y}[u_\eta]\eta_x + u_\eta\frac{\partial}{\partial y}[\eta_x]$$

now expand all of these $\frac{\partial}{\partial y}$'s. $(*)$

$$= \left[\frac{\partial}{\partial \xi}u_\xi \xi_y + \frac{\partial}{\partial \eta}u_\xi \eta_y\right]\xi_x + u_\xi \xi_{xy} + \left[\frac{\partial}{\partial \xi}u_\eta \xi_y + \frac{\partial}{\partial \eta}u_\eta \eta_y\right]\eta_x + u_\eta \eta_{xy}$$

hidden y's in $u(\xi,\eta)$ $\qquad$ $\xi = \xi(x,y)$ take deriv simply

$$= u_{\xi\xi}\xi_y\xi_x + u_{\xi\eta}\eta_y\xi_x + u_\xi\xi_{xy} + u_{\eta\xi}\xi_y\eta_x + u_{\eta\eta}\eta_y\eta_x + u_\eta\eta_{xy}$$

$$= u_{\xi\xi}\xi_y\xi_x + u_{\xi\eta}[\xi_x\eta_y + \xi_y\eta_x] + u_{\eta\eta}\eta_y\eta_x + u_\xi\xi_{xy} + u_\eta\eta_{xy} \checkmark$$

Last Time Canonical Forms... how do we come up with equations for ξ and η.

Introduce $\xi = \xi(x,y)$ $\eta = \eta(x,y)$
plug into general form.

Force certain coef derivatives to go to zero. For $u_{\xi\eta} = \phi(\ldots)$ need $u_{\xi\xi}$ and $u_{\eta\eta}$ to disappear.

This leads to
$$\left[\xi_x / \xi_y\right] = \frac{-B + \sqrt{B^2 - 4AC}}{2A}$$
$$\left[\eta_x / \eta_y\right] = \frac{-B - \sqrt{B^2 - 4AC}}{2A}.$$

Then thinking about coordinate systems... we seek characteristic curves
$$\xi(x,y) = C_1 \quad \text{and} \quad \eta(x,y) = C_2$$

This leads to:
$$\frac{dy}{dx} = -\left[\xi_x / \xi_y\right]$$
$$\frac{dy}{dx} = -\left[\eta_x / \eta_y\right]$$

We can always use this to rewrite HYPERBOLIC EQNS in canonical form
$$u_{\xi\eta} = \phi(\xi, \eta, u, u_\xi, u_\eta)$$

NOTE... need different formulation for other forms but similar Idea!

<u>EX</u> Consider the Hyperbolic Equation.

$$y^2 u_{xx} - x^2 u_{yy} = 0$$ and rewrite it completely in canonical form.

First Find The coordinate transformation.

$$\left[\xi_x / \xi_y\right] = \frac{+\sqrt{-4(y^2)(-x^2)}}{2y^2} = \frac{\sqrt{4x^2y^2}}{2y^2} = \frac{x}{y}$$

$$\left[\eta_x / \eta_y\right] = -\frac{x}{y}$$

then connect $\frac{dy}{dx}$ eqns.

$$\frac{dy}{dx} = -\frac{x}{y} \qquad \text{and} \qquad \frac{dy}{dx} = \frac{x}{y}$$

$$y\,dy = -x\,dx \qquad\qquad y\,dy = x\,dx$$

$$\tfrac{1}{2}y^2 = -\tfrac{1}{2}x^2 + c_1 \qquad\qquad \tfrac{1}{2}y^2 = \tfrac{1}{2}x^2 + c_2$$

$$x^2 + y^2 = \hat{c}_1 \qquad\qquad -x^2 + y^2 = \hat{c}_2$$

so let $\xi = y^2 - x^2$, $\eta = y^2 + x^2$ use these to transform the equation!

need u_{xx} and u_{yy}

$$u_x = -u_\xi 2x + u_\eta 2x = 2x(u_\eta - u_\xi)$$

$$u_y = 2y(u_\eta + u_\xi)$$

then $u_{xx} = \frac{\partial}{\partial x} u_x = \frac{\partial}{\partial x}\left[2x(u_\eta - u_\xi)\right]$

$$= 2(u_\eta - u_\xi) + 2x\frac{\partial}{\partial x}(u_\eta - u_\xi)$$

$$= 2(u_\eta - u_\xi) + 2x\left(-u_{\eta\xi}2x + u_{\eta\eta}2x + u_{\xi\xi}2x - u_{\eta\xi}2x\right)$$

$$= 2(u_\eta - u_\xi) + 4x^2(u_{\xi\xi} - 2u_{\eta\xi} + u_{\eta\eta})$$

$$u_{yy} = \frac{\partial}{\partial y}u_y = \frac{\partial}{\partial y}\left[2y(u_\eta + u_\xi)\right]$$

$$= 2(u_\eta + u_\xi) + 2y\frac{\partial}{\partial y}(u_\eta + u_\xi)$$

$$= 2(u_\eta + u_\xi) + 2y\left(+u_{\eta\xi}2y + u_{\eta\eta}2y + u_{\xi\xi}2y + u_{\xi\eta}2y\right)$$

$$= 2(u_\eta + u_\xi) + 4y^2(u_{\xi\xi} + 2u_{\eta\xi} + u_{\eta\eta})$$

plug in.

$$y^2\left[2(u_\eta - u_\xi) + 4x^2(u_{\xi\xi} - 2u_{\eta\xi} + u_{\eta\eta})\right] \mp x^2\left[2(u_\eta + u_\xi) + 4y^2(u_{\xi\xi} + 2u_{\eta\xi} + u_{\eta\eta})\right] = 0$$

what do we expect? All $u_{\xi\xi}$ and $u_{\eta\eta}$ should cancel out!!

Gather terms.

$$u_{\xi\xi}[4x^2y^2-4x^2y^2] + u_{\eta\xi}[-8x^2y^2-8x^2y^2] + u_{\eta\eta}[4x^2y^2-4x^2y^2]$$
$$+ u_\xi[-2y^2-2x^2] + u_\eta[2y^2-2x^2] = 0$$

$$u_{\eta\xi}[-16x^2y^2] - 2u_\xi[x^2+y^2] + 2u_\eta[y^2-x^2] = 0$$

$$u_{\eta\xi} = \frac{2[u_\xi(x^2+y^2) - u_\eta(y^2-x^2)]}{-16x^2y^2}$$

$$u_{\eta\xi} = \frac{-u_\xi(x^2+y^2) + u_\eta(y^2-x^2)}{8x^2y^2}$$ look good...

now get rid of x and y!

we know that $\xi = y^2 - x^2$
and $\eta = x^2 + y^2$

what about x^2y^2 to get cross terms
consider ξ^2 and η^2

$$\xi^2 = y^4 - 2x^2y^2 + x^4$$
$$\eta^2 = y^4 + 2x^2y^2 + x^4$$

then $\eta^2 - \xi^2 = 4x^2y^2$ or $x^2y^2 = \frac{1}{4}(\eta^2-\xi^2)$

so $$u_{\eta\xi} = \frac{-\eta u_\xi + \xi u_\eta}{2(\eta^2-\xi^2)}$$

We found our coordinates AND
we transformed into
canonical form.

Recall that the hyperbolic equation has two canonical forms!

SWITCH TO OTHER FORM $\longrightarrow$ let $\alpha = \xi + \eta$
$\beta = \xi - \eta$

to get $$u_{\alpha\alpha} - u_{\beta\beta} = \psi(\alpha, \beta, u, u_\alpha, u_\beta)$$
other form.

ASIDE

NOTE. You can derive this by forcing.

$$u_{\eta\xi} = \phi(\eta, \xi, u, u_\xi, u_\eta)$$

let $\alpha = \alpha(\xi, \eta)$
$\beta = \beta(\xi, \eta)$

plug into $u_{\eta\xi} \longrightarrow$

$$u_{\xi\eta} = \alpha_\xi \alpha_\eta u_{\alpha\alpha} + \beta_\eta \beta_\xi u_{\beta\beta} + [\alpha_\eta \beta_\xi + \alpha_\xi \beta_\eta] u_{\alpha\beta} + u_\alpha \alpha_{\eta\xi} + u_\beta \beta_{\xi\eta}.$$

then force $\alpha_\eta \beta_\xi + \alpha_\xi \beta_\eta = 0$

so that $u_{\alpha\beta}$ disappears.

$$\alpha_\xi \alpha_\eta = -\beta_\eta \beta_\xi$$

so that $u_{\alpha\alpha} - u_{\beta\beta}$ works on LHS.

SWITCH TO OTHER FORM. $\alpha(\xi, \eta) = \xi + \eta$, $\beta = \xi - \eta$.

$$\boxed{u_{\eta\xi} = \frac{-\eta u_\xi + \xi u_\eta}{2(\eta^2 - \xi^2)}}$$

$$u_\xi = \frac{\partial}{\partial \xi} u(\alpha, \beta) = u_\alpha \cancelto{1}{\alpha_\xi} + u_\beta \cancelto{1}{\beta_\xi}$$
$$= u_\alpha + u_\beta$$

$$u_\eta = \frac{\partial}{\partial \eta} u(\alpha, \beta) = u_\alpha \cancelto{1}{\alpha_\eta} + u_\beta \cancelto{-1}{\beta_\eta}$$
$$= u_\alpha - u_\beta$$

$$u_{\eta\xi} = \frac{\partial}{\partial \eta}[u_\xi] = \frac{\partial}{\partial \eta}[u_\alpha + u_\beta]$$
$$= u_{\alpha\alpha} \cancelto{1}{\alpha_\eta} + u_{\alpha\beta} \cancelto{-1}{\beta_\eta} + u_{\beta\alpha} \cancelto{1}{\alpha_\eta} + u_{\beta\beta} \cancelto{-1}{\beta_\eta}$$
$$= u_{\alpha\alpha} - u_{\beta\beta}$$

plug in

$$u_{\alpha\alpha} - u_{\beta\beta} = \frac{-\eta(u_\alpha + u_\beta) + \xi(u_\alpha - u_\beta)}{2(\eta^2 - \xi^2)}$$

$$= \frac{u_\alpha(\xi - \eta) - u_\beta(\xi + \eta)}{2(\eta - \xi)(\eta + \xi)} = \frac{\beta u_\alpha - \alpha u_\beta}{2(-\beta)(\alpha)}$$

Partial Differential Equations

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Homework and Reading - Day 23:

Due Thursday 12/5 - after break

Read and Take Notes

1. *Farlow* - Lesson 23.
2. *Farlow - read ahead* - You can read ahead of the class if you want. The next new chapter we will be covering is Lesson 25 on Finite Sine and Cosine Transforms.

Homework

1. *Problems* - Do problems all at the end of Chapter 23 - this means finish the problems that you already started on the last homework and continue with the new problems $4-6$.