DAY 24

Hand in Lessons 22/23 homework.

Today – skip to Lesson 27.

* You guys seem pretty comfortable with
Sep of Var
Eigenfunction Expand and
Integral Transform.

* Something totally new...

First Order Equations – Method of Characteristics

NOTE. This idea was on the putnam in 2011.

GENERAL FORM.

$$a(x,t)u_x + b(x,t)u_t + c(x,t)u = 0$$

Linear, Homogeneous, First Order.

Need ONE BC and ONE IC – sep of Var might work.
or $-\infty < x < \infty$. No Eigenfunctions!

PHYSICALLY – what does this model...

$$u_t = \alpha^2 u_{xx} - v u_x \longrightarrow \text{diffusion and convection.}$$

if $\alpha = 0$ then we have pure convection.
A concentration moving downstream with velocity v.

THEN $b(x,t)u_t = -a(x,t)u_x$
means convection where the speed changes depending on x and t.

$$v = a(x,t)/b(x,t)$$

The stream velocity varies.

<u>THINK</u> about a real stream... drop in some coloring The initial shape gets very distorted because of the varying velocities.

General Idea – Seek curves in the xt-plane along which our solution propagates (streamlines)

- if you dropped a single point into a stream and followed it ... it would travel along these lines, no diffusion.

* THESE ARE CHARACTERISTICS (or characteristic curves)

Introduce new coordinates:

s – changes along the characteristics
τ – changes along the initial curve. usually $t=0$

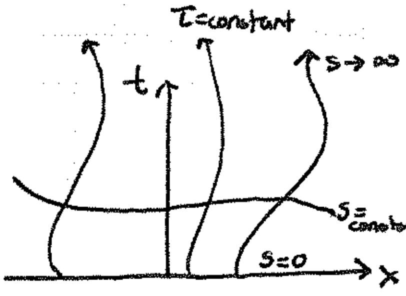
and transform our pde from:

$$a(x,t)u_x + b(x,t)u_t + c(x,t)u = 0$$

to an ode:

$$\frac{du}{ds} + c(x,t)u = 0$$

where $x = x(s)$ and $t = t(s)$



if we are attaching ourselves to move along This curve then x and t are functions of s.

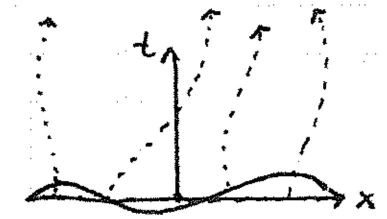
Then $$\frac{du}{ds} = \frac{du}{dx}\frac{dx}{ds} + \frac{du}{dt}\frac{dt}{ds} = u_x\frac{dx}{ds} + u_t\frac{dt}{ds}$$

$$= u_x a(x,t) + u_t b(x,t)$$

We choose our characteristics from

$$\frac{dx}{ds} = a(x,t) \qquad \frac{dt}{ds} = b(x,t).$$

$x(0) = \tau$ $\qquad$ $t(0) = 0$



The IC's attach us to t=0 and an initial disturbance.

Solution method:

1. Find the characteristics
2. Transform to an ODE in s and τ
   Solve the ODE
3. Transform back to x, and t.

<u>Ex</u> $u_x + u_t + 2u = 0 \qquad -\infty < x < \infty$

$u(x,0) = \sin(x)$ infinite domain.

What do we expect... $u_t = -u_x - 2u$

moving right

loosing energy

1. Find the characteristics...

$$\frac{dx}{ds} = 1 \quad \text{and} \quad \frac{dt}{ds} = 1$$

then $x = s + c_1$ and $t = s + c_2$

$x(0) = c_1 = \tau$ and $t(0) = c_2 = 0$

so $x = s + \tau$ and $t = s$

2. This transforms our PDE to and ODE

$$\frac{du}{ds} + 2u = 0$$

$u($[illegible]$0) = \sin(\tau)$

This is always the form by our definition $\frac{dx}{ds} = a \quad \frac{dt}{ds} = b$

You can do the transformation and double check.
let $s = t$ and $\tau = x - t$
then $u_x = u_\tau$.
$u_t = u_s - u_\tau$. ✓

Solving the ODE ...

$$u(s,\tau) = Ae^{-2s} = A(\tau)e^{-2s}$$

$$u(s,0) = A(\tau) = \sin(\tau)$$

$$u(s,\tau) = \sin(\tau)e^{-2s}.$$

3. Transform back

$s = t$ and $\tau = x - t$

$$u(x,t) = \sin(x-t)e^{-2t}$$

as expected a sine wave traveling to the right and damping to zero.

EX $x u_x + u_t + t u = 0 \qquad -\infty < x < \infty$

$u(x,0) = F(x)$

Find the characteristics:

$$\frac{dx}{ds} = x \qquad\qquad \frac{dt}{ds} = 1$$

$$\left.\begin{array}{ll} x = c_1 e^{s} & t = s + c_2 \\ x(0) = \tau & t(0) = 0 \\ x = \tau e^{s} & t = s \end{array}\right\rangle \quad \boxed{\begin{array}{l} s = t \\ \tau = x e^{-t} \end{array}}$$

Solve the ODE – can just write this down as long as you know why it works!

$$\left.\begin{array}{l} \dfrac{du}{ds} + s u = 0 \\ u(0) = F(\tau) \end{array}\right\}$$

seperable first order ode: $\dfrac{du}{u} = -s\,ds$

$$\ln|u| = -\tfrac{1}{2}s^2 + c(\tau)$$

$$u = A(t)e^{-s^2/2}$$

$$u(0) = A(t) = F(\tau)$$

$$u(s,\tau) = F(\tau)e^{-s^2/2}$$

Transform back

$$u(x,t) = F(xe^{-t})e^{-t^2/2}$$

for any given initial condition.

does the satisfy the PDE and ICs.

$$u(x,0) = F(xe^0)e^0 = F(x) \quad \checkmark$$

$$u_x = F'e^{-t}e^{-t^2/2} \qquad\qquad u = Fe^{-t^2/2}$$

$$u_t = -F'xe^{-t}e^{-t^2/2} + Fe^{-t^2/2}(-t)$$

plugging in

$$xe^{-t}\cancel{F'e^{-t^2/2}} - \cancel{xe^{-t}F'e^{-t^2/2}} - \cancel{tFe^{-t^2/2}} + tF\cancel{e^{-t^2/2}} = 0 \quad \checkmark$$

it works !!

This can be generalized to higher order!

$$a(x,y,t)\,u_x + b(x,y,t)\,u_y + c(x,y,t)\,u_t + d(x,y,t)\,u = 0$$

let
$$\frac{dx}{ds} = a(x,y,t) \qquad x(0) = \tau$$
$$\frac{dy}{ds} = b(x,y,t) \qquad y(0) = \eta$$
$$\frac{dt}{ds} = c(x,y,t) \qquad t(0) = 0$$

harder to visualize

but this is like surface waves.