Today we start talking about Fourier Transforms.

NOTE. A true understanding of Fourier Transforms requires Complex Analysis! So there are lots of things we will need to jump over.
Complex is not a pre-requsite.

* when doing our Eigenfunction Expansion we assumed

$$u(x,t) = \sum_{n=0}^{\infty} T_n(t) X_n(x)$$

where These were eigenfunctions, periodic, often sine and cosine.

DISCRETE...

Consider The Fourier Series...

$$f(x) = \frac{b_0}{2} + \sum_{n=1}^{\infty} b_n \cos\left(\frac{n\pi x}{L}\right) + a_n \sin\left(\frac{n\pi x}{L}\right)$$

looks like many of our eigenfunctions!

For us a Finite Domain. { Turns out we could assume this as our soln... any bounded periodic function That contains a finite # of max, min, and discontinuities can be represented by This.

This idea was behind why our fourier sine or cosine transforms work... we don't HAVE to use eigenfunctions.

One way That this is useful is in thinking about FREQUENCY SPACE.

- we are rewriting $f(x)$ in terms of sinusoids and The amount of each frequency describes "frequency space"

DISCRETE $\quad C_n = \sqrt{a_n^2 + b_n^2}$

Frequency Spectrum is a set of numbers $\{C_n\}$

Ex Expand $f(x) = \sin(\pi x)$ $[0,1]$ as Fourier Series.

$$\sin(\pi x) = b_0/2 + \sum_{n=1}^{\infty} b_n \cos(n\pi x) + a_n \sin(n\pi x)$$

orthogonality: $b_0 = 0$ $b_n = 0$ $a_1 = 1$ $a_n = 0$ $n \neq 1$

The discrete frequency spectrum $C_n = \sqrt{a_n^2 + b_n^2} = \begin{cases} 1 & n=1 \\ 0 & \text{otherwise} \end{cases}$

by transforming into frequency space we get insight into the function.

CONTINUOUS...

now we want to extend our ideas to a continuous frequency spectrum that works on $(-\infty, \infty)$.

The integral representation of this:

NOTE THE a_n B_n are now continuous.

$$f(x) = \int_0^{\infty} a(\xi)\cos(\xi x)\,d\xi + \int_0^{\infty} b(\xi)\sin(\xi x)\,d\xi$$

and we define $C(\xi) = \sqrt{a^2(\xi) + b^2(\xi)}$ — frequency is a function of ξ

continuous spectrum!

where now we allow $a(\xi)$ and $b(\xi)$ to contain complex numbers...

ASIDE... Complex #'s.

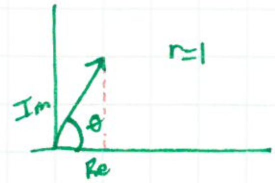
REVIEW really useful and elegant in applied math and physics $\longrightarrow$ TAKE COMPLEX ANALYSIS!

complex # $\quad z = x + iy \qquad \mathrm{Re}[z] = x \qquad \mathrm{Im}[z] = y$

x ← real part, iy ← imaginary part. and $i^2 = -1 \rightarrow i = \sqrt{-1}$

all complex #'s have a complex conjugate

$z^* = x - iy$ and the magnitude $|z| = \sqrt{z \cdot z^*} = \sqrt{x^2 + y^2}$



Ex $z = 1 + 2i$

$Re[z] = 1$

$Im[z] = 2$

$z^* = 1 - 2i$

$$|z| = \sqrt{z z^*} = \sqrt{\underbrace{(1+2i)(1-2i)}_{\text{FOIL}}} = \sqrt{1^2 - (2i)^2} = \sqrt{1+4} = \sqrt{5}$$

in polar form: $e^{i\theta} = \cos\theta + i\sin\theta$

and $z = |z| e^{i\theta}$

often think of these like vectors with real and imag components.

So the idea is that we want to describe things in terms of a continuous frequency spectrum ... this along with complex #'s and polar form allow us to write.

$$\mathcal{F}[f] = F(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-i\xi x} dx \qquad \text{Fourier Transform.}$$

$$\mathcal{F}^{-1}[F] = f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(\xi) e^{i\xi x} d\xi \qquad \text{Inverse Transform.}$$

in Physics ... sometimes this constant is $\frac{1}{2\pi}$ and 1

$F(\xi)$ is in frequency space ... resolve a function into frequencies.

NOTE ... These work for non-periodic functions BUT the improper integrals must converge. So many simple functions: [redacted] DO NOT have a fourier transform.

NEED:

$f(x) \to 0$ sufficiently fast as $|x| \to \infty$.

so $u \to 0$ and $u_x \to 0$ as $|x| \to \infty$

PROPERTIES OF THE TRANSFORM...

the inverse: $\mathcal{F}^{-1}\left[\mathcal{F}[f]\right] = f(x)$

linear $\mathcal{F}[af + bg] = a\mathcal{F}[f] + b\mathcal{F}[g]$

partial derivatives:

$$\mathcal{F}[u_x] = i\xi\mathcal{F}[u]$$

$$\mathcal{F}[u_{xx}] = -\xi^2\mathcal{F}[u]$$

$$\mathcal{F}[u_t] = \frac{d}{dt}\mathcal{F}[u]$$

$$\mathcal{F}[u_{tt}] = \frac{d^2}{dt^2}\mathcal{F}[u]$$

** Transform in space. **

You will prove these.

★ $\boxed{\text{Convolution:}}$ Multiplication does not work!! $\mathcal{F}[f(x)g(x)] \neq \mathcal{F}(f)\cdot\mathcal{F}(g)$

instead we have convolution

mult in Transform Space

convolution in "normal" space.

$$\underbrace{\mathcal{F}[f]\cdot\mathcal{F}[g]}_{\text{multiplication of the transforms}} = \underbrace{\mathcal{F}[f * g]}_{\text{convolution of the functions.}}$$

mult in transform space becomes convolution in x-space.

where $f * g = g * f = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x-\xi)g(\xi)\,d\xi.$ Formula.

we will use this in our final step of solving our PDEs. we must find the inverse transform. ✓

Imagine that...

$\Phi(\xi) = \mathcal{F}[\phi(x)]$ Initial condition.

Ex we find that

Soln in Transform Space

$$F(\xi) = \Phi(\xi)\cdot\frac{1}{\sqrt{2}}e^{-\xi^2/4}$$ as our PDE Soln. in transform space.

so our solution is $u(x,t) = \mathcal{F}^{-1}\left[\Phi(\xi)\cdot\frac{1}{\sqrt{2}}e^{-\xi^2/4}\right]$

on the tables This is e^{-x^2}!

we have:

$\mathcal{F}^{-1}[\mathcal{F}[f]\cdot\mathcal{F}[g]] = f * g$

take the transform and

so we will use convolution

convolved.

$$f(x) = \mathcal{F}^{-1}[\Phi(\xi)] * \mathcal{F}^{-1}\left[\frac{1}{\sqrt{2}}e^{-\xi^2/4}\right] = \Phi(x) * e^{-x^2}$$ IC.

Ex find $f * g$ if $f(x) = x$ and $g(x) = e^{-x^2}$

#1 plug into formula

#2 Evaluate integrals.

$$f * g = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}(x-\xi)e^{-\xi^2}\,d\xi = \frac{1}{\sqrt{2\pi}}\left[x\underbrace{\int_{-\infty}^{\infty}e^{-\xi^2}\,d\xi}_{=\sqrt{\pi}} - \frac{1}{\sqrt{2\pi}}\underbrace{\int_{-\infty}^{\infty}\xi e^{-\xi^2}\,d\xi}_{0}\right]$$

memorize formula

EVALUATE THESE →

How do we evaluate these ...

$$I_1 = \int_{-\infty}^{\infty} e^{-\xi^2} d\xi$$

substitution doesn't work ... but there is a trick ...

consider $I_1^2 = \underbrace{\int_{-\infty}^{\infty} e^{-x^2} dx}_{I_1} \underbrace{\int_{-\infty}^{\infty} e^{-y^2} dy}_{I_1} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-x^2-y^2} dx\, dy$

TWO DIFFERENT I_1's

turn this into a double integral.

now switch to polar $r^2 = x^2 + y^2$ $dxdy = rdrd\theta$

$$I_1^2 = \int_0^{2\pi}\int_0^{\infty} e^{-r^2} r\,dr\,d\theta$$

over all space

now we can do a substitution.

let $w = -r^2$
$dw = -2r\,dr$

$\lim_{w \to -\infty} e^{+w} = 0$

$$= \int_0^{2\pi}\int_0^{-\infty} -\frac{1}{2} e^{w} dw\, d\theta = \int_0^{2\pi} -\frac{1}{2} e^{w}\Big|_0^{-\infty} d\theta = \frac{1}{2}\int_0^{2\pi} d\theta = \pi$$

$I_1^2 = \pi \longrightarrow I_1 = \sqrt{\pi}$ so $\boxed{\int_{-\infty}^{\infty} e^{-\xi^2} d\xi = \sqrt{\pi}}$ ✓

memorize

$$I_2 = \int_{-\infty}^{\infty} \xi e^{-\xi^2} d\xi = \int_{-\infty}^{0} \xi e^{-\xi^2} d\xi + \int_0^{\infty} \xi e^{-\xi^2} d\xi$$

sub would give $\int_{-\infty}^{\infty} d\xi$ problem

sep.

let $w = -\xi^2$
$dw = -2\xi d\xi$

now substitution works.

$$= \int_{-\infty}^{0} -\frac{1}{2} e^{w} dw + \int_0^{-\infty} -\frac{1}{2} e^{w} dw = \frac{1}{2}\left[\int_{-\infty}^{0} e^{w} dw - \int_{-\infty}^{0} e^{w} dw\right] = 0$$

or $\frac{1}{2}\left[1 - \lim_{b\to-\infty} e^{b} - 1 + \lim_{b\to-\infty} e^{b}\right] = 0.$

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so $f * g = \frac{1}{\sqrt{2\pi}}\left[x\sqrt{\pi} - 0\right] = \frac{x}{\sqrt{2}}$ a new function of x.

From the integrals.

Finally $\longrightarrow$ Lets apply all of this to a PDE...

The Cauchy Problem

$$\begin{cases} u_t = \alpha^2 u_{xx} & -\infty < x < \infty \\ & 0 < t < \infty \\ u(x,0) = \phi(x) \end{cases}$$

<u>infinite</u> 1-D rod.
space $\Rightarrow$ fourier transform.

note we assume
$u \to 0$ as $|x| \to \infty$

Transform each piece...

$$\mathcal{F}[u_t] = F' \qquad \mathcal{F}[u_{xx}] = -\xi^2 F$$

by properties of derivatives.

$$\mathcal{F}[u(x,0)] = F(0) = \mathcal{F}[\phi(x)] = \Phi(\xi)$$

just the Transform of $\phi(x)$.

New Problem:

so
$$\left.\begin{aligned} F' &= -\alpha^2 \xi^2 F \\ F(0) &= \Phi(\xi) \end{aligned}\right\}$$

$F = F(\xi, t)$ but we imagine ξ as being constant.

solve the ODE. in time.

$$F(\xi,t) = A(\xi)\, e^{-\alpha^2 \xi^2 t}$$

$$F(\xi,0) = A(\xi) = \Phi(\xi) \longrightarrow F = \Phi(\xi)\, e^{-\alpha^2 \xi^2 t}$$

<u>mult</u> in transform space!

Find the inverse transform... because $\boxed{u(x,t) = \mathcal{F}^{-1}[F]}$

NOTE $\mathcal{F}^{-1}\left[e^{-\alpha^2 \xi^2 t}\right] = \dfrac{1}{\alpha\sqrt{2t}}\, e^{-(x^2/4\alpha^2 t)}$ From the tables! p.400 #6.

so we use convolution

$$u(x,t) = \mathcal{F}^{-1}\Big[\underbrace{\Phi(\xi)\, e^{-\alpha^2 \xi^2 t}}_{\text{mult.}}\Big] = \underbrace{\phi(x) * \frac{1}{\alpha\sqrt{2t}}\, e^{-(x^2/4\alpha^2 t)}}_{\text{convolution.}}$$

Note this almost always happens. don't take the time to transform IC.

$$\boxed{u(x,t) = \frac{1}{\sqrt{2\pi}} \cdot \frac{1}{\alpha\sqrt{2t}} \int_{-\infty}^{\infty} \phi(\xi)\, e^{-(x-\xi)^2/4\alpha^2 t}\, d\xi}$$

Green's Function or impulse response function.