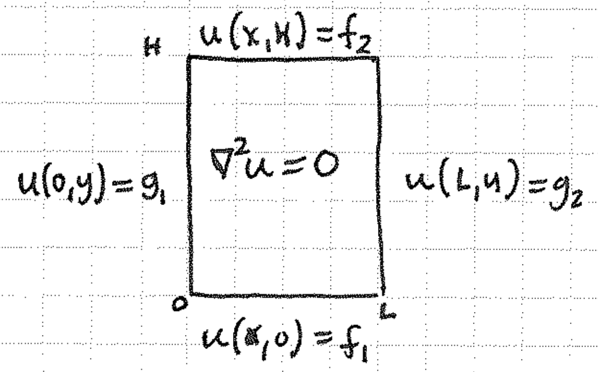
Day 8

More Practice with Separation of Variables.

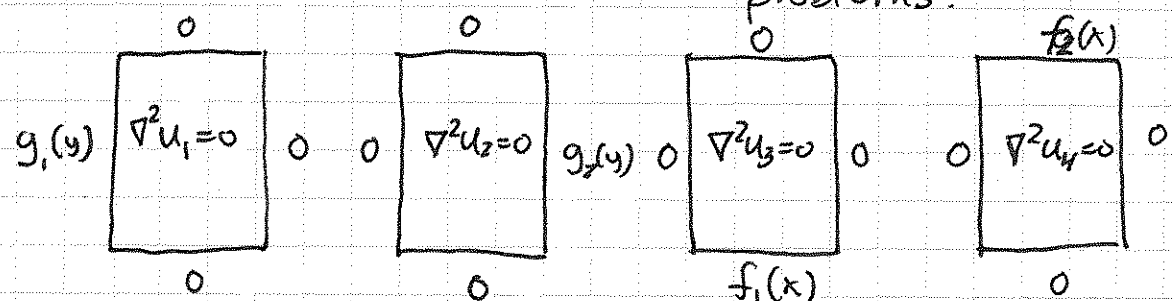
Laplace's Equation inside a Rectangle.



$\nabla^2 \to \dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2}$

Steady State heat Diffusion.
w/ homogeneous PDE
and non-homogeneous BC's.

To use separation of Vars we actually divide this into four problems:



if all boundaries were zero what would we expect?

so $u(x,y) = u_1 + u_2 + u_3 + u_4$ each one accounts separately for one of the boundaries.

similar to the heat eqn $u = S + T \ldots$

Choose one...

$$\left.\begin{aligned} &\nabla^2 u_1 = 0 \\ &u_1(0,y) = g_1(y) \\ &u_1(L,y) = 0 \\ &u_1(x,0) = 0 \\ &u_1(x,H) = 0 \end{aligned}\right\}$$

This problem is homogeneous in y and nonhomogeneous in x.

Do separation w/ eigenvalue problem in the homogeneous direction.

assume $u_1 = X(x)Y(y)$ then $\nabla^2 u = u_{xx} + u_{yy}$

$$X''Y + Y''X = 0 \quad \text{divide by } XY$$

$$\frac{X''}{X} + \frac{Y''}{Y} = 0 \quad \text{or} \quad \frac{X''}{X} = \frac{-Y''}{Y} = K.$$

$X'' - KX = 0$

$X(L) = 0$

can't separate the other condition.

not really a boundary value problem.

Solve this second...

and

$Y'' + KY = 0$

$Y(0) = 0$

$Y(H) = 0$

This is an eigenvalue problem

- Here what do we expect for K... from experience solving this BVP?

assume $K = +\lambda^2$

$Y'' + \lambda^2 Y = 0$

$\lambda = 0$ then $Y(y) = Ay + B$ $\quad Y(0) = B = 0 \quad Y(H) = AH = 0 \quad A = 0$

TRIVIAL

$\lambda > 0$ then $Y(y) = A\sin(\lambda y) + B\cos(\lambda y)$

$Y(0) = B = 0$

$Y(H) = A\sin(\lambda H) \quad \lambda H = n\pi \quad \lambda = \frac{n\pi}{H}$

This should be very familiar.

$Y_n(y) = A_n \sin\left(\frac{n\pi y}{H}\right)$ and $K = +\left(\frac{n\pi}{H}\right)^2$

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now solve the X problem.

$X'' - \left(\frac{n\pi}{H}\right)^2 X = 0 \quad r = \pm \frac{n\pi}{H} \quad n = 1, 2, 3, \ldots$ real distinct roots.

$X(x) = Ae^{\frac{n\pi x}{H}} + Be^{\frac{-n\pi x}{H}}$ apply the BC... a little messy

$X(L) = Ae^{n\pi L/H} + Be^{-n\pi L/H} = 0$

so $A = -Be^{-2n\pi L/H}$

$X(x) = +B\left[e^{-n\pi x/H} - e^{+n\pi/H(x-2L)}\right] = A\sinh\left(\frac{n\pi}{H}(x-L)\right)$

(*)

HOW DID WE GET THIS...

$\sinh(x) = \frac{e^x - e^{-x}}{2}$ so we can rewrite (*) as

$B\left[e^{-n\pi x/H} - e^{n\pi x/H} e^{-2n\pi L/H}\right] = Be^{-n\pi L/H}\left[e^{\frac{-n\pi x}{H}} e^{n\pi L/H} - e^{\frac{n\pi x}{H}} e^{\frac{-n\pi L}{H}}\right]$

$= Be^{-n\pi L/H}\left[e^{-n\pi/H(x-L)} - e^{\frac{n\pi}{H}(x-L)}\right] = \underbrace{-2Be^{-n\pi L/H}}_{\text{just a constant.}}\left[\frac{e^{\frac{n\pi}{H}(x-L)} - e^{\frac{-n\pi}{H}(x-L)}}{2}\right]$

$= A\sinh\left(\frac{n\pi}{H}(x-L)\right)$

So our solution is

$$\boxed{u_1(x,t) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi y}{H}\right) \sinh\left(\frac{n\pi}{H}(x-L)\right)}$$

and we use the remaining condition to solve for the A_n's.

$$u_1(0,y) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi y}{H}\right) \sinh\left(\frac{n\pi}{H}(-L)\right) = g_1(y)$$

mult by $\sin\left(\frac{m\pi y}{H}\right)$ and integrate.

$$\int_0^H \sum_{n=1}^{\infty} A_n \sinh\left(\frac{n\pi}{H}(-L)\right) \sin\left(\frac{n\pi y}{H}\right) \sin\left(\frac{m\pi y}{H}\right) dy = \int_0^H g_1(y) \sin\left(\frac{m\pi y}{H}\right) dy$$

zero unless $m=n$

from orthogonality

$$A_n \sinh\left(\frac{n\pi}{H}(-L)\right) \frac{H}{2} = \int_0^H g_1(y) \sin\left(\frac{m\pi y}{H}\right) dy$$

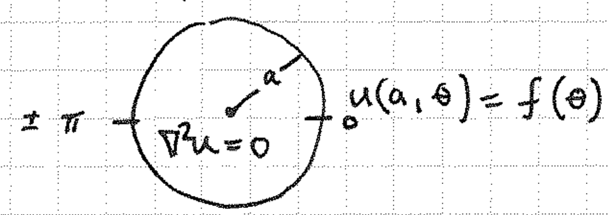
so

$$\boxed{A_n = \frac{2}{H} \cdot \frac{1}{\sinh\left(\frac{n\pi}{H}(-L)\right)} \int_0^H g_1(y) \sin\left(\frac{m\pi y}{H}\right) dy.}$$

Then we solve for u_2, u_3, and u_4 in a similar way to get our full soln.

1. Separate the BC's so that all but one is homogeneous.
2. Assume $u = X(x)Y(y)$ sub in ... you should get one eigenvalue problem the other is missing one BC.
3. Solve the eigenvalue problem
4. Solve the other ODE combine constants.
5. Write down the general soln
6. Use the final BC to solve for the constants orthogonality!

Laplaces equation on a Circular Disk.



now we use polar coordinates!

$$\nabla^2 \to \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}$$

$$\left.\begin{aligned} \nabla^2 u &= \frac{1}{r}\frac{\partial}{\partial r}\left(r\, u_r\right) + \frac{1}{r^2}u_{\theta\theta} = 0 \\ u(a,\theta) &= f(\theta) \end{aligned}\right\}$$

how do we deal with the fact that we don't have enough boundary conditions?

Similar to the circular rod...
at $\pm\pi$ force continuity: $u(r,-\pi) = u(r,\pi)$
$u_\theta(r,-\pi) = u_\theta(r,\pi)$

What do we assume at $r=0$ ... physically we assume that $|u(0,\theta)| < \infty$

The temperature cannot go to infinity at the center.
"Solutions are bounded" common assumption.

Lets go ahead and separate variables...

Assume $u = R(r)\,\Theta(\theta)$ plug this in.

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r R'\Theta\right) + \frac{1}{r^2}R\Theta'' = 0$$

$$\frac{1}{r}\left[R'\Theta + rR''\Theta\right] + \underbrace{\frac{1}{r^2}R\Theta''}_{} = 0$$

divide by $\frac{R\Theta}{r^2}$

$$\frac{r}{R}\left[R' + rR''\right] + \frac{\Theta''}{\Theta} = 0$$

$$\frac{r}{R}\left[R' + rR''\right] = -\frac{\Theta''}{\Theta} = K.$$

separated!

TWO ODEs one for $R(r)$
the other for $\Theta(\theta)$.

$$\left.\begin{array}{l}\frac{r}{R}\left[R' + rR''\right] = K \\ |R(0)| < \infty\end{array}\right\}$$

$$\left.\begin{array}{l}-\Theta'' = K\Theta \\ \Theta'' + K\Theta = 0 \\ \Theta(-\pi) = \Theta(\pi) \\ \Theta'(-\pi) = \Theta'(\pi)\end{array}\right\}$$

which is an eigenvalue problem?

Solve this first!

we solved this last time!

$K = \lambda^2$

$\Theta(\theta) = A_n \sin(n\theta) + B_n \cos(n\theta)$

$\lambda^2 = n^2 \quad n = 0, 1, 2 \ldots$

Look up class notes

$L = \pi$

Then solve the "non-eval" problem:

$$\frac{r}{R}\left[R' + rR''\right] = n^2$$

From ODEs...

$$rR' + r^2R'' = n^2R$$

$$r^2R'' + rR' - n^2R = 0$$

Linear ODE with non constant coefficients

Equidimensional ODE ... special form

notice the independent var multiplies in same order as derivative.

Here we assume solutions of the form

$R(r) = r^k$ powers of r

$R'(r) = kr^{k-1}$

$R''(r) = k(k-1)r^{k-2}$

plug in.

$$r^2k(k-1)r^{k-2} + rkr^{k-1} - n^2r^k = 0$$

$$k(k-1)r^k + kr^k - n^2r^k = 0 \quad \text{divide by } r^k$$

$$k(k-1) + k - n^2 = 0 \quad \text{charicteristic eqn for } k.$$

$$k^2 - n^2 = 0 \qquad k = \pm n$$

For $n \neq 0$ we have two distinct roots:

so $\quad R(r) = Ar^n + Br^{-n}$

Our only condition is $|R(0)| < \infty \Rightarrow B = 0$

$$R(r) = Ar^n$$

For $n = 0$ special case repeater roots $\quad R(r) = A + B\ln(r) \quad$ again $|R(0)| < \infty$ so $B = 0$

$R(r) = A$ arbitrary constant still works!

putting our soln back together:

$$\boxed{u(r,\theta) = \sum_{n=0}^{\infty} A_n r^n \cos(n\theta) + \sum_{n=1}^{\infty} B_n r^n \sin(n\theta)}$$

now we use the condition at $r=a$ to solve for A_n and B_n
just like last time there are two calculations:

1. Orthogonality of $\cos(m\theta)$
   with $n=0$ and $n \neq 0$
2. Orthogonality of $\sin(m\theta)$

$$A_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)\, d\theta$$

$n > 0$

$$A_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta) \cos(n\theta)\, d\theta$$

$$B_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta) \sin(n\theta)\, d\theta.$$

Partial Differential Equations

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Homework and Reading - Day 8:

Tuesday 10/1 - hand in Thursday 10/2 (??)

Read and Take Notes

1. *Additional Chapter Online* - SOV for Laplace's Equation - This section contains both problems that we solved in class. Also, it has a pretty cool discussion about the Mean Value Theorem, The Maximum Principle, and a proof that Laplace's equation is both Well Posed and has Unique Solutions. (FUN!!!)
2. *Read Ahead* - Lesson 9 - Farlow - Solving Nonhomogeneous PDEs using Eigenfunction Expansion. We will start this on Tuesday 10/2

Homework

1. *Homework Problems*
   1. Laplace's Equation on the unit square, centered at origin, with one non-homogeneous boundary condition:

$$\begin{aligned}
&\nabla^2 u = 0\\
&u(-1,y) = 0\\
&u(1,y) = 0\\
&u(x,-1) = 0\\
&u(x,1) = \cos(\pi x)
\end{aligned}$$

   please show all the steps! Explain in words what you are doing with each step.

   2. Laplace's Equation on the unit square with a flux: (note: this square is not centered at the origin)

$$\begin{aligned}
&\nabla^2 u = 0\\
&u_x(0,y) = 0\\
&u_x(L,y) = 0\\
&u(x,0) = 0\\
&u(x,H) = f(x)
\end{aligned}$$

   3. What is the solution to the completely homogeneous Laplace's Equation:

$$\begin{aligned}
&\nabla^2 u = 0\\
&u(-1,y) = 0\\
&u(1,y) = 0\\
&u(x,-1) = 0\\
&u(x,1) = 0
\end{aligned}$$

   You can use your work from problem 1. Does the solution make physical sense? Why?

4. Solve Laplace's Equation inside the semicircle $0 < r < a$ and $0 < \theta < \pi$, where the base of the semicircle is kept at zero temperature and the top arch is at a prescribed temperature. Remember to use the polar coordinate form of the Lapacian ∇^2.

$$
\begin{aligned}
&\nabla^2 u = 0 \\
&u(a, \theta) = g(\theta) \\
&u(r, 0) = 0 \\
&u(r, \pi) = 0
\end{aligned}
$$

What is the additional condition that we require for $u(r, \theta)$ when $r = 0$?

NOTE: General Solution: $u(r, \theta) = \sum_{n=1}^{\infty} B_n r^n \sin(n\theta)$