Eigen-value problems... Differential equations where information is supplied at the BOUNDARY!

"Boundary Value Problems"

<u>Ex</u>

$$\left.\begin{aligned} X'' + \lambda^* X &= 0 \\ X(0) &= 0 \\ X(L) &= 0 \end{aligned}\right\}$$

assume $\lambda \in \mathbb{R}$

Generally λ^* is some physical constant that can take on a variety of values.

- Much more restrictive to specify boundaries
- New part of the problem $\longrightarrow$ solving possible values of λ.

1. Solve the Second order Linear ODE.
   Characteristic Eqn: $r^2 + \lambda^* = 0 \qquad r = \pm\sqrt{-\lambda}$

   the soln depends on whether λ is POS, NEG, or ZERO

CASES:

$\lambda < 0$ then $-\lambda$ is positive AND $r = \pm\sqrt{|\lambda|} = \pm\lambda^*$

$$X(x) = Ae^{\lambda^* x} + Be^{-\lambda^* x}$$

APPLY BC's

$$X(0) = A + B = 0, \quad \text{so} \quad A = -B$$

$$X(L) = Ae^{\lambda^* L} + Be^{-\lambda^* L} = 0$$

$$-Be^{\lambda^* L} + Be^{-\lambda^* L} = 0$$

$$B\left(e^{-\lambda^* L} - e^{\lambda^* L}\right) = 0$$

either $B = 0$ <u>or</u> $e^{-\lambda^* L} = e^{\lambda^* L}$

TRIVIAL SOLN
$A = -B = 0$
$X(x) = 0$

$-\lambda^* L = \lambda^* L = 0$

$L = 0$ or $\lambda = 0$

$\downarrow \qquad \downarrow$

NO! $\qquad$ NO!

Contradiction.

FOR THE BOUNDARY part of the problem... THESE NEVER WORK!!

$\lambda < 0$ problems w/ physical and mathematical reality.

OFTEN WE SOLVE

$$X'' + \alpha^2 X = 0$$

$\uparrow$

to avoid having to test $\lambda < 0$ case.

$\lambda = 0$ then we have a repeated root $r=0$

$$X(x) = Ax + B$$

APPLY BC's

$$\left.\begin{aligned} X(0) &= B = 0 \\ X(L) &= AL = 0 \quad A=0 \end{aligned}\right\} \quad \text{TRIVIAL SOLN } X(x) = 0$$

$\lambda > 0$ then $-\lambda$ is negative and $r = \pm i\sqrt{\lambda} = \pm i\lambda^*$

$$X(x) = A\sin(\lambda^* x) + B\cos(\lambda^* x)$$

APPLY BC's.

$$X(0) = B = 0$$
$$X(L) = A\sin(\lambda^* L) = 0$$

either $A=0$ or $\sin(\lambda^* L) = 0$

TRIVIAL SOLN
NO!

WAIT! $\sin(\lambda^* L) = 0$ works when $\lambda^* L = n\pi$

or

$$\lambda^* = \frac{n\pi}{L} > 0 \quad n = 1, 2, 3$$

So we have a family of possible solns...

$$X(x) = A\sin\left(\frac{n\pi}{L} x\right)$$

we are left w/ this constant because we have specified λ instead of A.

A is specified by other physical conditions
NORMALIZATION OR INITIAL CONDITIONS.

EIGENFUNCTION : $\sin\left(\frac{n\pi x}{L}\right) = \psi_n$

EIGENVALUE : $\lambda_n = \sqrt{\frac{n\pi}{L}}$

* Solns change drastically depending on the Boundary Conditions!!

How is this used in PDEs...

Lesson 5 Farlow

Separation of Variables (SOV)

1. Use for Linear Homogeneous PDEs
2. Boundary conditions must also be Homogeneous

$$\underbrace{\begin{aligned} \alpha u_x(0,t) + \beta u(0,t) &= 0 \\ \gamma u_x(L,t) + \beta u(L,t) &= 0 \end{aligned}}_{\text{Linear w/ } \alpha,\ \beta,\ \gamma \text{ and } \delta \text{ constants.}} \Bigg\}$$

if there are functions or constants on this side you must transform the BC's first.

Ex

$$\begin{cases} u_t = \alpha^2 u_{xx} & 0 < x < 1 \quad 0 < t < \infty \\ u(0,t) = 0 & 0 < t < \infty \\ u(1,t) = 0 & \\ u(x,0) = \phi(x) & 0 \le x \le 1 \end{cases}$$

This is the first PDE we will solve!

#1. <u>Assume</u> $u(x,t) = X(x)T(t)$

u can be written as a function of only x times a function of only y.

#2 Sub this into our PDE

$$XT' = \alpha^2 T X''$$

#3 Rearange the PDE so there are only X's on one side and only T's on the other

$$\frac{T'}{T} = \frac{\alpha^2 X''}{X} \longrightarrow \frac{T'}{\alpha^2 T} = \frac{X''}{X}$$

NOTE ⟶ keep the constant w/ the T's because the boundary X'' eqn is an eigenvalue eqn.

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We have a function of only t = a function of only x then the both must equal a constant because x and t are independent variables.

$$\frac{T'}{\alpha^2 T} = \frac{X''}{X} = K$$

constant... it will turn into an eigenvalue.

#5 Breake the PDE into two ODE's and solve

$$\frac{X''}{X} = K \qquad \Bigg\| \qquad \frac{T'}{\alpha^2 T} = K$$

solve the ODEs

$$X'' - KX = 0 \qquad \Bigg\| \qquad T' = \alpha^2 K T$$
$$X'' - KX = 0 \qquad \Bigg\| \qquad T' - \alpha^2 K T = 0$$

seperable IVP

$$T(t) = Ae^{K\alpha^2 t}$$

because of this we need $K<0$ otherwise our soln $\to \infty$ as $t \to \infty$. Doesn't make physical sense for solns to grow.

assume

Force $K<0$ $\rightarrow$ $K = -\lambda^2$

$$X'' + \lambda^2 X = 0$$

$$X(x) = A\sin\lambda x + B\cos\lambda x$$

so $u(x,t) = e^{-\lambda^2\alpha^2 t}\left[A\sin\lambda x + B\cos\lambda x\right]$

Technically we have a soln now we must apply the Boundary and Initial Conditions.

In practice skip this.

But really I always

APPLY THE BC's FIRST...

$$u(0,t) = X(0)T(t) = 0$$
$$u(1,t) = X(1)T(t) = 0$$

we know that $T(t) \neq 0$

#6 APPLY BOUNDARY CONDITIONS.

Apply BC's to the eigenval problem.

$$X(0) = 0$$
$$X(1) = 0$$

$$X(0) = B = 0 \qquad X(1) = A\sin\lambda = 0$$

the separation constant is an eigenvalue.

so $\lambda_n = \pm n\pi \quad n = 1,2,3 \ldots$

Now we have an Infinite # of functions that would satisfy our PDE and our BC's

$$u_n(x,t) = A_n e^{-(n\pi\alpha)^2 t}\sin(n\pi x) \qquad n = 1,2,3,\ldots$$

These are sometimes called fundamental Solns.

Constants all absorbed into one.

For a Linear Homogeneous PDE the sum of solutions is also a soln ...

#7 General Soln Before Initial Conditions

$$u(x,t) = \sum_{n=1}^{\infty} A_n e^{-(n\pi\alpha)^2 t} \sin(n\pi x)$$

is our general soln before applying the IC.

APPLY THE INITIAL CONDITION

$$u(x,0) = \phi(x)$$

so $$\phi(x) = \sum_{n=1}^{\infty} A_n \sin(n\pi x) \quad (\star)$$ use orthogonality to solve this.

this is just saying that we expand our Initial condition into a fourier sine series.

OUR JOB ... FIND the A_n's.

The functions sin and cos are orthogonal

$$\int_0^1 \sin(m\pi x)\sin(n\pi x)\,dx = \begin{cases} 0 & m \neq n \\ 1/2 & m = n \end{cases}$$

MULTIPLY BOTH SIDES OF $(\star)$ BY $\sin(m\pi x)$ AND INTEGRATE ... this gets rid of the sum so you can solve for the A_n's.

$$\int_0^1 \phi(x)\sin(m\pi x)\,dx = \int_0^1 \sum_{n=1}^{\infty} A_n \sin(n\pi x)\sin(m\pi x)\,dx$$

$$\int_0^1 \phi(x)\sin(m\pi x)\,dx = A_m \int_0^1 \sin^2(m\pi x)\,dx$$

$$= \frac{A_m}{2}$$

$$A_m = 2\int_0^1 \phi(x)\sin(m\pi x)\,dx$$

so given a $\phi(x)$ we just need to integrate to find the A_n's.

We just solved all PDEs of the form.

$$\begin{cases} u_t = \alpha^2 u_{xx} & 0<x<1 \quad 0<t<\infty \\ u(0,t)=0 & \\ u(1,t)=0 & 0<t<\infty \\ u(x,0)=\phi(x). & 0\le x\le 1 \end{cases}$$

integrating eigen functions ...
over their domain...

NOTES.

$$\int_0^1 \sin^2(m\pi x)\,dx = \frac{1}{2}\int_0^1 [1-\cos(2m\pi x)]\,dx$$

$$= \frac{1}{2}\left[x - \frac{1}{2m\pi}\underbrace{\sin(2m\pi x)}_{0}\right]\Bigg|_0^1 = \frac{1}{2}.$$

expect nice #'s.

Partial Differential Equations

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Homework and Reading - Day 3:

Due Thursday 9/12

Read and Take Notes

1. *Farlow* - Lesson 5
2. *Farlow - read ahead* - You can read ahead of the class if you want. The next chapter we will be covering is Lesson 6 on transforming non-homogeneous boundary conditions into homogeneous ones.

Homework - Due Thursday 9/12

1. *Farlow* - Do all problems at the end of Lesson 5.
2. *Additional Problem - Challenge* - Use Separation of variables to solve the following PDE:

$$U_t = \alpha^2 U_{xx}; \;\; U(0,t) = 0, \; U_x(1,t) = 0; \;\; U(x,0) = \phi(x)$$

   Similar to what we did in class you will end up with a family of solutions and a condition on the A_n constants that depends on an integral.