DAY 4.

Hand in HW 2 and 3

FARLOW lesson 6
Transforming Nonhomogeneous BC's into Homogeneous ones.

IF your boundary conditions are of the form

$$\alpha_1 u_x(0,t) + \beta_1 u(0,t) = g_1(t)$$
$$\alpha_2 u_x(L,t) + \beta_2 u(L,t) = g_2(t)$$

and $g_1(t) \neq 0$, $g_2(t) \neq 0$

Then you want to transform the pde so that the BC's are homogeneous.

Then solve.

- IF the PDE is homogeneous after transformation then separation of variables will work
- IF the PDE is nonhomogeneous after transformation then you must use other methods.

Consider the equation

$$\begin{cases} u_t = \alpha^2 u_{xx} & 0 < x < L \quad 0 < t < \infty \\ u(0,t) = k_1 & 0 < t < \infty \\ u(L,t) = k_2 & \\ u(x,0) = \phi(x) & 0 \le x \le L \end{cases}$$

Because k_1 and k_2 are nonzero we will not be able ~~to~~ to separate the boundary conditions.

OUR GOAL ... see if we can ~~[illegible]~~ transform this problem so the BC's are homogeneous.

First ... what is the steady state solution

$$\left.\begin{aligned} u_{xx} &= 0 \\ u(0) &= k_1 \\ u(L) &= k_2 \end{aligned}\right\} \quad \begin{aligned} u &= Ax + B \\ u(0) &= B = k_1 \\ u(L) &= AL + k_1 = k_2 \\ A &= \frac{k_2 - k_1}{L} \end{aligned}$$

$$u = \left(\frac{k_2 - k_1}{L}\right) x + k_1$$ a straight line.

What would the soln for steady state be in the homogeneous case

$u(0)=0 \quad u(L)=0 \quad \Rightarrow \quad u \equiv 0$
also a straight line.

<u>AND</u> other than this long term ... the short term solution would decrease basically the same way.

Boundary conditions effect the long term steady state ...

SO WE BREAK OUR SOLUTION INTO TWO PARTS ...

$$u(x,t) = \underbrace{\text{steady state}}_{\text{long time soln}} + \underbrace{\text{transient}}_{\text{part of the soln that depends on the initial conditions.}}$$

1. Solve the steady state

2. Assume: $u(x,t) = \underbrace{\left(\frac{k_2 - k_1}{L}\right) x + k_1}_{\text{this is the steady state we just solved for}} + U_E(x,t)$

Sub this into our pde ... $\left.\begin{aligned} u_t &= V_t \\ u_{xx} &= V_{xx} \end{aligned}\right\}$ the steady state part is removed in derivatives.

$$\left.\begin{array}{c} u_t = \alpha^2 u_{xx} \\ \downarrow \qquad \downarrow \\ U_t = \alpha^2 U_{xx} \end{array}\right\} \boxed{\text{PDE}}$$

no change to the PDE

$$\boxed{\text{BC's}} \begin{cases} u(0,t) = k_1 + U(0,t) = k_1 \\ \quad \text{so } U(0,t) = 0 \\ u(L,t) = k_2 + U(L,t) = k_2 \\ \quad \text{so } U(L,t) = 0 \end{cases}$$

this is what we wanted ! our BC's are homogeneous.

$\boxed{\text{IC}}$

$$u(x,0) = \left(\frac{k_2 - k_1}{L}\right)x + k_1 + U(x,0) = \phi(x)$$

$$\text{so} \quad U(x,0) = \phi(x) - \left(\frac{k_2-k_1}{L}\right)x - k_1$$

our non homogeneous term was transfered to the IC's

OKAY to have complicated IC's.

The transformed PDE Becomes:

$$\begin{cases} U_t = \alpha^2 U_{xx} \\ U(0,t) = 0 \\ U(L,t) = 0 \\ U(x,0) = \phi(x) - \left(\frac{k_2-k_1}{L}\right)x - k_1 \end{cases}$$

This is a homogeneous pde w/ homogeneous BC's so we can use separation of Variables.

NOTE: Same PDE and BC that we solved before...

$$U(x,t) = \sum_{n=1}^{\infty} A_n e^{-\alpha^2\frac{(n\pi)^2}{L^2}t} \sin(n\pi x/L)$$

use orthogonality to solve for the A_n's

$$\int_0^L \left(\phi(x) - \frac{(k_2-k_1)}{L}x - k_1\right)\sin\left(\frac{m\pi x}{L}\right)dx$$

$$= \underbrace{\int_0^L \sum_{n=1}^{\infty} A_n \sin(n\pi x/L)\sin(m\pi x/L)\,dx}_{}$$

this is zero except when $m=n$

Integrate.

$$\int_0^L A_n \sin^2\left(\frac{m\pi x}{L}\right)dx = \frac{A_n}{2}\int_0^L \left(1 - \cos\left(\frac{2m\pi x}{L}\right)\right) dx$$

$$= \frac{A_n}{2}\left[x - \frac{L}{2m\pi}\cancelto{0}{\sin\left(\frac{2m\pi x}{L}\right)}\right]_0^L$$

$$= \frac{A_n L}{2}$$

so

$$\int_0^L \left[\phi(x) - \left(\frac{k_2 - k_1}{L}\right)x - k_1\right]\sin\left(\frac{n\pi x}{L}\right)dx = \frac{A_n L}{2}$$

or

$$A_n = \frac{2}{L}\int_0^L \left[\phi(x) - \left(\frac{k_2-k_1}{L}\right)x - k_1\right]\sin\left(\frac{n\pi x}{L}\right)dx$$

NOTE — the extra $1/L$ term comes from integrating from 0 to L rather than 0 to 1 like before in lesson 5.

IN GENERAL FOR CONSTANT NONHOMOGENEOUS PARTS OF BC

$$A_n = \frac{2}{L}\int_0^L \underbrace{\Phi(x)}_{\text{new initial condition}}\, \underbrace{U(x,0)}_{\text{transient soln.}}\, dx$$

Stop... Questions.

NEXT WE CONSIDER NON CONSTANT / TIME VARYING BC's.

$$\begin{cases} u_t = \alpha^2 u_{xx} & 0 < x < L \quad 0 < t < \infty \\ u(0,t) = g_1(t) & \\ & 0 < t < \infty \\ u_x(L,t) + h\,u(L,t) = g_2(t) & \\ u(x,0) = \phi(x) & \end{cases}$$

Similar Idea. We use a general form for our steady state ...

We seek solutions of the form

This looks strange but works! Compare to the steady state.

$$u(x,t) = \underbrace{A(t)\left[1 - x/L\right] + B(t)\left[x/L\right]}_{\bar{u}} + U(x,t) \qquad (*)$$

- this is just a guess we hope that it will satisfy the BC's and transform them to Homogeneous in the U equation.

OUR FIRST JOB: Find $A(t)$ and $B(t)$ such that the first part of $(*)$ satisfies the BC's.

Apply BC's just to $\bar{u}$...

$u(0,t) = g_1(t)$

$u_x(L,t) + hu(L,t) = g_2(t)$. plug in and solve.

want $\bar{u}(0,t) = g_1(t) \Rightarrow A(t) = g_1(t)$

want $\bar{u}_x(L,t) + h\,\bar{u}(L,t) = g_2(t)$

Solving for $A(t)$ and $B(t)$.

$$\Rightarrow -\frac{A(t)}{L} + \frac{B(t)}{L} + h\,B(t) = g_2(t)$$

$$-A(t) + B(t)\left[1 + hL\right] = Lg_2(t)$$

$$B(t) = \frac{Lg_2(t) + A(t)}{\left[1 + hL\right]} = \frac{Lg_2(t) + g_1(t)}{\left[1 + hL\right]}$$

so $$u(x,t) = \overset{A(t)}{g_1(t)}\left[1 - x/L\right] + \overset{B(t)}{\frac{g_1(t) + Lg_2(t)}{\left[1 + hL\right]}}\left[x/L\right] + U(x,t)$$

sub this into the PDE to get

$$u_{xx} = U_{xx} \quad \text{but} \quad u_t = S_t + U_t$$

time derivative not so simple!

where S_t is the derivative of all the $g_1(t)$ $g_2(t)$ terms.

Our BC's become homogeneous... we already know this...

Initial Condition...

$$u(x,0) = S(x,0) + \mathcal{U}(x,0) = \phi(x)$$

$$\mathcal{U}(x,0) = \phi(x) - S(x,0)$$

where $S(x,0)$ is the term containing $g_1(0)$ and $g_2(0)$.

So our transformed PDE Becomes.

$$\begin{cases} \mathcal{U}_t = \alpha^2 \mathcal{U}_{xx} - S_t & 0 < x < L \quad 0 < t < \infty \\ \mathcal{U}(0,t) = 0 & 0 < t < \infty \\ \mathcal{U}_x(L,t) + h\mathcal{U}(L,t) = 0 & \\ \mathcal{U}(x,0) = \phi(x) - S(x,0) & 0 < x < L \end{cases}$$

when $S_t \equiv 0$ this is homogeneous and we can use separation of variables

when $S_t \neq 0$ we must use other methods.