Another Example — Transforming Boundary Conditions...

$$\left.\begin{aligned} &u_t = u_{xx} \\ &u(0,t) = \sin(t) \\ &u(1,t) = 0 \\ &u(x,0) = 1 \end{aligned}\right\}$$ Homogeneous PDE w/ Non-homogeneous Time Dependent Boundary Conditions.

$$u(x,t) = \text{"steady state"} + \underset{v(x,t)}{\text{Transient}}$$

"SOLVE" STEADY STATE

For time dep. <u>Assume!</u> $S = A(t)\left[1 - x/L\right] + B(t)\left[x/L\right]$

plug into BC's. $L=1$

APPLY BC's $\longrightarrow$ PLUG INTO BCS.

$S(0,t) = A(t) = \sin(t)$

$S(1,t) = B(t) = 0$

so $S(x,t) = \sin(t)[1-x]$ this should always satisfy the BCs.

Assume $u(x,t) = S(x,t) + v(x,t) = \sin(t)[1-x] + v(x,t)$

PDE $$\left.\begin{aligned} u_t &= \cos(t)[1-x] + v_t \\ u_{xx} &= v_{xx} \end{aligned}\right\rangle \quad v_t = v_{xx} - \cos(t)[1-x]$$

BC: $u(0,t) = \overset{\sin(t)}{S(0,t)} + v(0,t) = \sin(t)$ $\quad v(0,t) = 0$

$u(1,t) = S(1,t) + v(1,t) = 0$ $\quad v(1,t) = 0$

As Expected.

IC: $u(x,0) = S(x,0) + v(x,0) = 1$

$= 0 + v(x,0) = 1$ $\quad v(x,0) = 1$

NEW PDE

$$\left.\begin{aligned} &v_t = v_{xx} - \cos(t)[1-x] \\ &v(0,t) = 0 \\ &v(1,t) = 0 \\ &v(x,0) = 1 \end{aligned}\right\}$$ Non-homogeneous PDE w/ homogeneous BCs.

Another Example — Separation of Variables. (Harder one)

$$\left.\begin{aligned} &u_t = \alpha^2 u_{xx} \\ &u(0,t) = 0 \\ &u_x + hu(1,t) = 0 \\ \\ &u(x,0) = x. \end{aligned}\right\}$$

homogeneous PDE w/ homogeneous BC's

SOV will work.

Assume $u(x,t) = X(x)T(t)$.

Do the full eigenvalue problem.

FARLOW LESSON 7.
Solving more Complicated Problems by Separation of Variables.

Ex. Heat-Flow Problem with Derivative BC

$$\begin{cases} u_t = \alpha^2 u_{xx} \\ u(0,t) = 0 \\ u_x(1,t) + h\,u(1,t) = 0 \\ u(x,0) = x \end{cases}$$

Let's solve this PDE w/ Separation of Variables.

General soln $u(x,t) = e^{-(\lambda\alpha)^2 t}\left(A\sin(\lambda x) + B\cos(\lambda x)\right)$

Now we apply BC's:

$$u(0,t) = A\sin(0) + B\cos(0) = 0$$
$$\Rightarrow B = 0$$

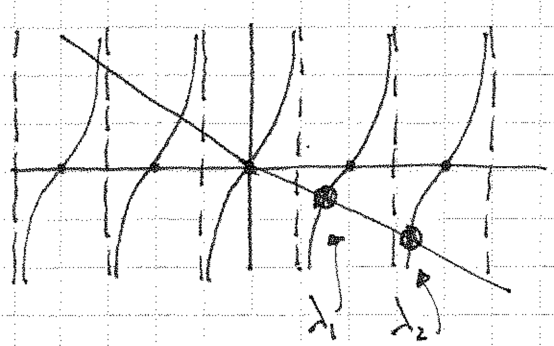
$$u_x(x,t) = e^{-(\lambda\alpha)^2 t}\left[\lambda A\cos(\lambda x)\right]$$

$$u_x(1,t) + h\,u(1,t) = e^{-(\lambda\alpha)^2 t}\left[\lambda A\cos(\lambda)\right] + h e^{-(\lambda\alpha)^2 t}\left[A\sin(\lambda)\right] = 0$$

$$\lambda\cos(\lambda) + h\sin(\lambda) = 0$$
$$\frac{\sin(\lambda)}{\cos(\lambda)} = -\frac{\lambda}{h}$$
$$\tan(\lambda) = -\frac{\lambda}{h}$$

this is a close as we can get to an exact soln for our eigenvalues!

Ex Solving for these types of e-vals
If $h=1$ then $\tan(\lambda) = -\lambda$ plot the two functions + look for intersections



λ can't be zero or less than zero

λ_m = numerically solved intersection points.

in this case:

n	λ_n
1	2.02
2	4.91
3	7.98
4	11.08
5	14.20

we can only find numerical zeros ... no closed form soln.

So we write our solution as

$$u(x,t) = \sum_{n=1}^{\infty} A_n e^{-\lambda_n^2 \alpha^2 t} \sin(\lambda_n x)$$

where λ_n satisfies $\tan(\lambda_n) = -\frac{\lambda_n}{h}$.

Now we find the A_n's by applying the IC's.

$$u(x,0) = x = \sum_{n=1}^{\infty} A_n \sin(\lambda_n x)$$

multiply by $\sin(\lambda_m x)$ and integrate across our domain.

$$\int_0^1 x \sin(\lambda_m x)\,dx = \int_0^1 \sum_{n=1}^{\infty} A_n \sin(\lambda_n x) \sin(\lambda_m x)\,dx$$

NOTE... Sturm and Liouville proved that the solns to eqns of the form

$$\begin{cases} [p(x)y']' - q(x)y + \lambda r(x) y = 0 \\ \alpha_1 y(0) + \beta_1 y'(0) = 0 \\ \alpha_2 y(1) + \beta_2 y'(1) = 0 \end{cases}$$

If we let
$p(x) = 1$
$q(x) = 0$
$\lambda r(x) = \lambda^2 \alpha^2$
Then we get our $X(x)$ eqn.
So the boundary part of the problem is a Sturm-Liouville problem.

have an infinite sequence of eigenvalues

$\lambda_1 < \lambda_2 < \dots < \lambda_n < \to \infty$

each eigenvalue has a corresponding non zero eigenfunction $y_n(x)$

the eigenfunctions are orthogonal

$$\int_0^1 r(x) y_n(x) y_m(x)\,dx = 0 \quad \text{when } m \neq n$$

we can use orthogonality

$$\int_0^1 x \sin(\lambda m x)\,dx = Am \int_0^1 \sin^2(\lambda m x)\,dx =$$

$$Am \int_0^1 \tfrac{1}{2}(1-\cos(2\lambda m x))\,dx = \frac{Am}{2}\left[x - \frac{1}{2\lambda m}\sin(2\lambda m x)\right]_0^1$$

$$= \frac{Am}{2}\left[1 - \frac{1}{2\lambda m}\sin(2\lambda m)\right]$$

Here we used $\sin^2(x) = \tfrac{1}{2}(1-\cos(2x))$

$$= \frac{Am}{2}\left[1 - \frac{1}{\lambda m}\sin(\lambda m)\cos(\lambda m)\right]$$

$\sin(2x) = 2\sin(x)\cos(x).$

$$= Am\left[\frac{\lambda m - \sin(\lambda m)\cos(\lambda m)}{2\lambda m}\right]$$

solving for the constant:

$$Am = \frac{2\lambda m}{\lambda m - \sin(\lambda m)\cos(\lambda m)} \int_0^1 x \sin(\lambda m x)\,dx.$$

so to solve for these we pick a λm then solve for Am.

Ex. if $h=1$

n	λm	A_n
1	2.02	0.24
2	4.91	0.22
3	7.98	-0.03
4	11.09	-0.11

$$u(x,t) = \sum_{n=1}^{\infty} A_n e^{-(\lambda m a)^2 t} \sin(\lambda m x)$$

We can write out the first few terms...

$$u(x,t) = (0.24)e^{-(2.02a)^2 t}\sin(2.02x) + (0.22)e^{-(4.91a)^2 t}\sin(4.91x) - 0.03e^{-(7.98a)^2 t}\sin(7.98x) \ldots$$