DAY 6

Lesson 8

Transforming Hard Equations into Easier Ones.

- As long as an equation is linear and homogeneous we can separate variables ... it just might take a bit more work.

Ex

$$\begin{cases} u_t = \alpha^2 u_{xx} - \beta u \\ u(0,t) = 0 \\ u(1,t) = 0 \\ u(x,0) = \phi(x) \end{cases}$$

Transformations often based on intuition about what the soln will look like.

lets consider parts of the problem...

If $\alpha = 0$ then $u_t = -\beta u$

solve like an ode ... first order seperable

$$\frac{\partial u}{\partial t} = -\beta u \qquad \frac{\partial u}{u} = -\beta \partial t \qquad \ln|u| = -\beta t + C$$

$$u = Ae^{-\beta t} \quad \text{technically } A = A(x)$$

but the general form has exponential damping.

So we try a transformation: $\boxed{u(x,t) = e^{-\beta t} w(x,t)}$

it's like we are breaking it up into a ~~[illegible]~~ diffusion part and a convection part.

SUB IN: (Term-by-term)

$$u_t = \frac{\partial}{\partial t}\left(e^{-\beta t} w(x,t)\right) = -\beta e^{-\beta t} w + e^{-\beta t} w_t$$

$$u_{xx} = \frac{\partial}{\partial x}\left[\frac{\partial}{\partial x}\left[e^{-\beta t} w(x,t)\right]\right] = e^{-\beta t} w_{xx}$$

$$u(0,t) = e^{-\beta t} w(0,t) = 0 \Rightarrow w(0,t) = 0$$
$$u(1,t) = e^{-\beta t} w(1,t) = 0 \Rightarrow w(1,t) = 0$$
$$u(x,0) = w(x,0) = \phi(x)$$

The transformed PDE:

$$-\beta e^{-\beta t} w + e^{-\beta t} w_t = \alpha^2 w_{xx} e^{-\beta t} - \beta e^{-\beta t} w$$

divide through by $e^{-\beta t}$
and cancel $-\beta e^{-\beta t} w$ terms
that appear on both sides.

$$\begin{cases} w_t = \alpha^2 w_{xx} \\ w(0,t) = 0 \\ w(1,t) = 0 \\ w(x,0) = \phi(x) \end{cases}$$

We achieved our goal! Our PDE is reduced to a simple heat equation... one <u>we have solved</u> before.

start a cheat sheet!

$$w(x,t) = \sum_{n=1}^{\infty} A_n e^{-(n\pi\alpha)^2 t} \sin(n\pi x)$$

$$A_n = 2\int_0^1 \phi(x) \sin(n\pi x)\,dx$$

Full Soln: $$u(x,t) = e^{-\beta t} \sum_{n=1}^{\infty} A_n \sin(n\pi x) e^{-(n\pi\alpha)^2 t}$$

$$A_n = 2\int_0^1 \phi(x) \sin(n\pi x)\,dx$$

<u>NOTES</u> A similar transformation for the diffusion-convection eqn

$$u_t = \alpha^2 u_{xx} - V u_x$$

assume: $$\boxed{u(x,t) = e^{V(x - Vt/2)/2\alpha^2} w(x,t)}$$

WHY... if $\alpha = 0$ then we have $u_t = -V u_x$
this is the "baby" wave equation
so we would look for traveling wave type solns and "guess and check" until we found a good transformation.

do an example.

Classify the PDE ... are there any pieces that you would like to get rid of? For the diffusion convection equation we use the transformation

$$u(x,t) = e^{V(x - Vt/2)/2\alpha^2} w(x,t)$$

if needed...

STEP 1 — GET RID OF NON-H BC's $u = u_s + \hat{u}$ OR GET RID of extra terms of RHS.

Ex
$$\left.\begin{aligned} u_t &= u_{xx} - u_x \\ u(0,t) &= 0 \\ u(1,t) &= 0 \\ u(x,0) &= e^{x/2} \end{aligned}\right\}$$

STEP 1 TRANSFORM IF NEEDED.

here $\alpha = 1$ and $V = 1$

so $u(x,t) = e^{\frac{1}{2}(x - t/2)} w(x,t)$

look this up or make a guess...

plug into the pde:

calculate each piece.

$$u_t = -\frac{1}{4} e^{\frac{1}{2}(x - t/2)} w(x,t) + e^{\frac{1}{2}(x-t/2)} w_t(x,t)$$

$$u_{xx} = \frac{1}{4} e^{\frac{1}{2}(x - t/2)} w(x,t) + e^{\frac{1}{2}(x-t/2)} w_x(x,t) + e^{\frac{1}{2}(x-t/2)} w_{xx}(x,t)$$

$$u_x = \frac{1}{2} e^{\frac{1}{2}(x - t/2)} w(x,t) + e^{\frac{1}{2}(x-t/2)} w_x(x,t)$$

let $s = \frac{1}{2}(x - t/2)$ for now ... because we are lazy.

plug into PDE

$$\cancel{-\tfrac{1}{4} e^s w} + e^s w_t = \cancel{\tfrac{1}{4} e^s w} + \cancel{e^s w_x} + e^s w_{xx} - \cancel{\tfrac{1}{2} e^s w} - \cancel{e^s w_x}$$

$$e^s w_t = e^s w_{xx} \quad \text{or} \quad w_t = w_{xx}$$

pure diffusion.

plug into BC's

just plug in

$$u(0,t) = e^{-t/4} w(0,t) = 0 \qquad w(0,t) = 0$$
$$u(1,t) = e^{\frac{1}{2}(1 - t/2)} w(1,t) = 0 \qquad w(1,t) = 0$$

this would be more complicated for nonhomogeneous

plug into IC's

$$u(x,0) = e^{x/2} w(x,0) = e^{x/2} \quad \text{so} \quad w(x,0) = 1$$

how nice of them!

STEP 2
WRITE DOWN
NEW PDE

$$\left.\begin{aligned} &w_t = w_{xx} \\ &w(0,t) = 0 \\ &w(1,t) = 0 \\ &w(x,0) = 1 \end{aligned}\right\}$$

I always re-write my pde to make sure I have all the pieces.
Make sure you have PDE, BC's and IC here.

re-assess where you are ... do we need more transformations?

STEP 3
SOLVE

→ the goal is to get the PDE into SOV form.
We have solved lots of these so we just know the soln here ... I will go through the steps one more time :)

SOV

(1.) Assume $w = T(t)X(x)$ and plug into pde

Review seperation of variables.

$$T'X = X''T \Rightarrow \frac{T'}{T} = \frac{X''}{X} = \text{constant}$$

(2.) Separate into two ODE's and solve

SPACE $\frac{X''}{X} = \lambda$ AND $\frac{T'}{T} = \lambda$ TIME

$X'' - \lambda X = 0$ SOLVE SPACE FIRST w/ BC's.

$X(0) = 0$
$X(1) = 0$ these math the original PDE *

$T' = \lambda T$ SOLVE TIME SECOND

Characteristic eqn $r^2 - \lambda = 0$

Eigenvalue problem!

$r = \pm\sqrt{\lambda} \Rightarrow$ $\lambda > 0$ real ✗ this almost never works!
$\lambda = 0$ repeated
$\lambda < 0$ imag.

$\lambda = 0$ $X(x) = Ax + B$
$X(0) = B = 0$ TRIVIAL.
$X(1) = A = 0$ always check this.

$\lambda \neq 0$ then $r = \pm i\sqrt{|\lambda|}$ or let $\lambda = -\alpha^2$ $r = \pm i\alpha$
$X(x) = A\sin\alpha x + B\cos\alpha x$
$X(0) = B = 0$
$X(1) = A\sin\alpha = 0 \Rightarrow \alpha = n\pi \quad n = 1, 2, 3 \ldots$

SPACE $\lambda = -(n\pi)^2$ $X_n = \sin(n\pi x)$

$T' = \lambda T$ $r = \lambda$ so $T(t) = Ae^{\lambda t} = Ae^{-(n\pi)^2 t}$

$$T_n(t) = Ae^{-(n\pi)^2 t}$$

③ Put the pieces back together

$$w(x,t) = T(t)\,X(x)$$ this was our SOV assumption.

$$w(x,t) = \sum_{n=1}^{\infty} A_n e^{-(n\pi)^2 t} \sin(n\pi x)$$

the soln is a sum of solns.

this is the sum of all possible solutions ... most general soln ... principle of superposition.

Because we transformed the IC ...

④ Apply Initial conditions to find A_n's. $w(x,0) = 1$ plug in $t=0$

Use the ideas from Fourier Series and Orthogonality.

$$1 = \sum_{n=1}^{\infty} A_n \sin(n\pi x)$$

use orthogonality to solve for A_n's.

$$\int_0^1 1\cdot \sin(m\pi x)\,dx = \int_0^1 \sum_{n=1}^{\infty} A_n \sin(n\pi x)\sin(m\pi x)\,dx$$

when in doubt just do the integrals on both sides and solve for constants.

just evaluate this.

$$= \frac{-1}{m\pi}\cos(m\pi x)\Big|_0^1$$

$$= \frac{-1}{m\pi}\left((-1)^m - 1\right) = \begin{cases} \frac{2}{m\pi} & m=\text{odd} \\ 0 & m=\text{even} \end{cases}$$

orthogonality $\Rightarrow$ $m=n$

$$\int_0^1 A_n \sin^2(n\pi x)\,dx = \frac{A_n L}{2}$$

but $L=1$

$$= \frac{A_n}{2}$$

LHS = RHS $\longrightarrow$ so $A_n = \begin{cases} \frac{4}{m\pi} & m=\text{odd} \\ 0 & m=\text{even} \end{cases}$ or $\frac{2}{n\pi}$ $m=2n$ $n=1,2,3\ldots$

⑤ Write final SOV solution

$$\boxed{w(x,t) = \sum_{n=1}^{\infty} \frac{2}{n\pi} e^{-(n\pi)^2 t} \sin(n\pi x)}$$

Use sum notation for final soln!

STEP 4
RECONSTRUCT
FINAL SOLN

If we did a transformation then we need to write down the soln to the untransformed problem.

$$u(x,t) = e^{\frac{1}{2}(x - t/2)} \sum_{n=1}^{\infty} \frac{2}{n\pi} e^{-(n\pi)^2 t} \sin(n\pi x)$$

and we are done!

if you do more than one transformation you go backward step by step ...

Aside:

Ex. $u(x,t) = u_s + \hat{u}$ $\quad u_s = x + 4$

and $\hat{u} = e^{-\beta t} w(x,t)$

and $w(x,t) = 4 + \sum_{n=1}^{\infty} e^{t} \sin(x)$

Put it back together ...

$$u(x,t) = x + 4 + e^{-\beta t}\left[4 + \sum_{n=1}^{\infty} e^{t} \sin(x)\right]$$

Treat each of the pieces of the soln as a separate problem ... then piece things together at the end.