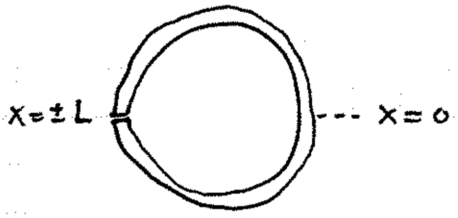
PDEs - Day 7. - Review

- Today we will pause and check for understanding. Lots of example problems...

Ex 1 Heat Conduction in a Thin Insulated Circular Ring.



Total Length = 2L

$$\begin{cases} u_t = \alpha^2 u_{xx} \\ u(-L,t) = u(L,t) \\ u_x(-L,t) = u_x(L,t) \\ u(x,0) = \phi(x) = 1+x \end{cases}$$

assume continuity! (for $u(-L,t) = u(L,t)$ and $u_x(-L,t) = u_x(L,t)$)

Physically - what do we expect?

- In the long term it should reach some constant temperature.

1. Classify: Linear, Second Order, Homogeneous PDE and BCs, ↗ why? Heat Eqn.
   - Try Separation of Variables.

2. Assume $u = T(t)X(x)$ plug in to PDE and BC's.

$$T'X = \alpha^2 T X'' \quad \text{so} \quad \frac{T'}{\alpha T} = \frac{X''}{X} = K$$

$T' = K\alpha T$

$T(t) = Ae^{\alpha^2 KT}$

expect decay not growth

$K \leq 0$

$k = -\lambda^2$

$X'' = -\lambda^2 X$ or $X'' + \lambda^2 X = 0$

Boundaries:

$\cancel{T(t)}X(-L) = \cancel{T(t)}X(L)$

assume $T(t) \neq 0$

$X(-L) = X(L)$

and $X'(-L) = X'(L)$ periodic BC's.

3. Solve the Eigenvalue Problem:

$$\left.\begin{aligned} X'' + \lambda^2 X = 0 \\ X(-L) = X(L) \\ X'(-L) = X'(L) \end{aligned}\right\}$$

Characteristic $r = \pm\sqrt{-\lambda^2}$

assume $\lambda = 0$ then $r = 0$ repeated root and $X(x) = Ax + B$.

now apply BC's.

$$X(-L) = X(L) \Rightarrow -AL + B = AL + B$$

$$-A = A \Rightarrow A = 0$$

$$X'(-L) = X'(L) \Rightarrow A = A \text{ or } 0 = 0 \quad \checkmark$$

$$\boxed{\psi_0 = B_0 \quad \lambda_0 = 0}$$

assume $\lambda > 0$ then $r = \pm i\lambda$ imag. and $X(x) = A\sin(\lambda x) + B\cos(\lambda x)$

now apply BC's.

$$X(-L) = X(L) \Rightarrow A\sin(-\lambda L) + B\cos(-\lambda L) = A\sin(\lambda L) + B\cos(\lambda L)$$

$$-A\sin(\lambda L) + \cancel{B\cos(\lambda L)} = A\sin(\lambda L) + \cancel{B\cos(\lambda L)}$$

$$2A\sin(\lambda L) = 0$$

works when $\lambda L = n\pi$

A_n is undefined. $\lambda = \frac{n\pi}{L}$

$$X'(-L) = X'(L) \Rightarrow \lambda A\cos(-\lambda L) - \lambda B\sin(-\lambda L) = \lambda A\cos(\lambda L) - \lambda B\sin(\lambda L)$$

$$\cancel{\lambda A\cos(\lambda L)} + \lambda B\sin(\lambda L) = \cancel{\lambda A\cos(\lambda L)} - \lambda B\sin(\lambda L)$$

$$2\lambda B\sin(\lambda L) = 0$$

works when $\lambda L = n\pi$

B_n is undefined. $\lambda = \frac{n\pi}{L}$

So any combination of B_0, $B_n\cos\left(\frac{n\pi}{L}x\right)$, $A_n\sin\left(\frac{n\pi}{L}x\right)$ works as a soln.

$$X(x) = B_0 + \sum_{n=1}^{\infty} B_n\cos\left(\frac{n\pi x}{L}\right) + \sum_{n=1}^{\infty} A_n\sin\left(\frac{n\pi x}{L}\right)$$

4. The general soln:

$$u(x,t) = B_0 + \sum_{n=1}^{\infty} B_n\cos\left(\frac{n\pi x}{L}\right)e^{-\left(\frac{n\pi\alpha}{L}\right)^2 t} + \sum_{n=1}^{\infty} A_n\sin\left(\frac{n\pi x}{L}\right)e^{-\left(\frac{n\pi\alpha}{L}\right)^2 t}$$

5. Now apply the Initial condition... and use orthogonality of eigenfunctions we will use the properties:

$$\left[\int_{-L}^{L} \cos\left(\frac{n\pi x}{L}\right)\cos\left(\frac{m\pi x}{L}\right)dx = \begin{cases} 0 & m \neq n \\ L & m = n \neq 0 \\ 2L & m = n = 0 \end{cases} \right]$$

$$\left[\begin{array}{l}\displaystyle\int_{-L}^{L} \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right)dx = \begin{cases} 0 & n \neq m \\ L & n = m \neq 0 \end{cases} \\[2em] \displaystyle\int_{-L}^{L} \sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{m\pi x}{L}\right)dx = 0 \end{array}\right]$$

This is just lots of integration

$$u(x,0) = \sum_{n=0}^{\infty} B_n \cos\left(\frac{n\pi x}{L}\right) + \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{L}\right) = 1 + x$$

note B_0 is absorbed here

Two calculations ... mult by $\cos\left(\frac{m\pi x}{L}\right)$ integrate $\int_{-L}^{L}$

FIRST ↗

$$\int_{-L}^{L} \sum_{n=0}^{\infty} B_n \cos\left(\frac{n\pi x}{L}\right)\cos\left(\frac{m\pi x}{L}\right)dx + \underset{\text{orth.}}{0} = \int_{-L}^{L} (1+x)\cos\left(\frac{m\pi x}{L}\right)dx$$

zero except $n=m$

$$\int_{-L}^{L} B_n \cos^2\left(\frac{n\pi x}{L}\right)dx = \int_{-L}^{L} (1+x)\cos\left(\frac{m\pi x}{L}\right)dx$$

when $n = m = 0$

$$\int_{-L}^{L} B_0 \, dx = \int_{-L}^{L} (1+x)\,dx \Rightarrow 2L B_0 = \left[x + \frac{1}{2}x^2\right]\Big|_{-L}^{L}$$

Do the integrals and use orthogonality.

$$2LB_0 = 2L \Rightarrow \boxed{B_0 = 1}$$

when $n = m \neq 0$

$$\int_{-L}^{L} B_n \cos^2\left(\frac{n\pi x}{L}\right)dx = \int_{-L}^{L} (1+x)\cos\left(\frac{m\pi x}{L}\right)dx$$

$u = 1+x \quad dv = \cos \, dx$

$du = 1dx \quad v = \frac{L}{m\pi}\sin\left(\frac{m\pi x}{L}\right)$

$$B_n L = \frac{(1+x)L}{m\pi}\cancelto{0}{\sin\left(\frac{m\pi x}{L}\right)}\Bigg|_{-L}^{L} - \frac{L}{m\pi}\int_{-L}^{L} \sin\left(\frac{m\pi x}{L}\right)dx$$

$$= \frac{L^2}{(m\pi)^2}\cos\left(\frac{m\pi x}{L}\right)\Bigg|_{-L}^{L} = \frac{L^2}{(n\pi)^2}\left[\cos(m\pi) - \cos(-m\pi)\right]$$

$$= 0$$

$$\boxed{B_n = 0}$$

now mult by $\sin\left(\frac{m\pi x}{L}\right)$ integrate $\int_{-L}^{L}$

SECOND ↗

$$\emptyset + \int_{-L}^{L} \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{m\pi x}{L}\right) dx = \int_{-L}^{L} (1+x) \sin\left(\frac{m\pi x}{L}\right) dx$$

zero except
$n=m$

$$\int_{-L}^{L} A_n \sin^2\left(\frac{n\pi x}{L}\right) dx = \int_{-L}^{L} (1+x) \sin\left(\frac{n\pi x}{L}\right) dx$$

$$\underline{\underline{A_n L}} = \int_{-L}^{L} (1+x) \sin\left(\frac{n\pi x}{L}\right) dx$$

$u = 1+x \quad dv = \sin(\ )dx$
$du = dx \quad v = \frac{-L}{m\pi}\cos\left(\frac{n\pi x}{L}\right)$

$$= -\frac{(1+x)L}{n\pi} \cos\left(\frac{n\pi x}{L}\right)\Big|_{-L}^{L} + \frac{L}{n\pi}\int_{-L}^{L} \cos\left(\frac{n\pi x}{L}\right) dx$$

$$= -\frac{(1+L)L}{n\pi}\cos(n\pi) + \frac{(1-L)L}{n\pi}\cos(-n\pi) + \left(\frac{L}{n\pi}\right)^2 \cancelto{0}{\sin\left(\frac{n\pi x}{L}\right)}\Big|_{-L}^{L}$$

$$= \frac{L\cos(n\pi)}{n\pi}\left[-1 - L + 1 - L\right] = \underline{\underline{\frac{-2L^2}{n\pi}\cos(n\pi)}}$$

$$\text{so} \quad A_n = \frac{-2L}{n\pi}\cos(n\pi)$$

6. Write down final soln:

$$\boxed{u(x,t) = 1 - \sum_{n=1}^{\infty} \frac{2L}{n\pi}\cos(n\pi)\sin\left(\frac{n\pi x}{L}\right) e^{-\left(\frac{n\pi\alpha}{L}\right)^2 t}}$$

Does this match your physical intuition? YES.

- as $t \to \infty$ the temperature becomes uniformly distributed.
- The total heat in the rod is preserved.

initially total heat $\int_{-L}^{L} \phi(x)\,dx = 1$

as $t \to \infty \quad \int_{-L}^{L} u(x,t)\,dx = 1.$

Ex 2. Nonhomogeneous Boundaries with heat loss along the lateral.

$$\left.\begin{array}{l} u_t = \alpha^2 u_{xx} - 2u \\ u(0,t) = 1 \\ u_x(1,t) = 1 \\ u(x,0) = \phi(x) \end{array}\right\}$$

Goal: get this into a simple form for Separation of Variables.

First deal with the heat loss then the boundaries.

1. Transform out lateral heat loss:

<u>assume</u> $\boxed{u(x,t) = e^{-2t} w(x,t)}$ plug into PDE...

$$u_t = -2e^{-2t} w + e^{-2t} w_t$$
$$u_{xx} = e^{-2t} w_{xx}$$

$$-2\cancel{e^{-2t}} w + e^{-2t} w_t = \alpha^2 e^{-2t} w_{xx} - 2\cancel{e^{-2t}} w$$

convection term cancels.

$$w_t = \alpha^2 w_{xx}$$

$$u(0,t) = e^{-2t} w(0,t) = 1 \qquad w(0,t) = e^{2t}$$
$$u_x(1,t) = e^{-2t} w_x(1,t) = 1 \qquad w_x(1,t) = e^{2t}$$

$$u(x,0) = w(x,0) = \phi(x)$$

NEW PDE

$$\left.\begin{array}{l} w_t = \alpha^2 w_{xx} \\ w(0,t) = e^{2t} \\ w_x(1,t) = e^{2t} \\ w(x,0) = \phi(x) \end{array}\right\}$$

homogeneous PDE with time dependent boundary conditions.

2. Transform out the nonhomogeneous BC's

<u>assume</u> $w = S + v$
steady + transient

where $S(x,t) = A(t)[1-x] + B(t)[x]$

Apply Boundary conditions to this ↺

$$S(0,t) = A(t) = e^{2t}$$
$$S_x(1,t) = -A(t) + B(t) = -e^{-2t} + B(t) = e^{2t}$$
$$B(t) = 2e^{2t}$$

$$S(x,t) = (1-x)e^{2t} + 2xe^{2t} = e^{2t} + xe^{2t} = (1+x)e^{2t}$$

my transformation: $\boxed{w = (1+x)e^{2t} + v(x,t)}$

plug into PDE...

$$w_t = (1+x)2e^{2t} + v_t$$
$$w_{xx} = v_{xx}$$
$$w(0,t) = e^{2t} + v(0,t) = e^{2t} \qquad v(0,t) = 0$$
$$w_x(1,t) = e^{2t} + v_x(1,t) = e^{2t} \qquad v_x(1,t) = 0$$

$$w(x,0) = v(x,0) + (1+x) = \phi(x)$$

$$v(x,0) = \phi(x) - (1+x)$$

NEW PDE

$$\left.\begin{aligned} &v_t = \alpha^2 v_{xx} - 2(1+x)e^{2t} \\ &v(0,t) = 0 \\ &v_x(1,t) = 0 \\ &v(x,0) = \phi(x) - (1+x) \end{aligned}\right\}$$

Now we have homogeneous BC's but a non-homog. PDE...

Need another method to solve this !!!

Final soln: $u(x,t) = e^{-2t}\left((1+x)e^{2t} + v(x,t)\right)$

Partial Differential Equations

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HOMEWORK AND READING - DAY 7:

THURSDAY 9/26

READ AND TAKE NOTES

1. *Farlow* - REVIEW - Chapters 5-8. Make sure you are comfortable with these ideas. This week we are focusing on our understanding of separation of variables.
2. *Additional Chapter Online* - SOV for Laplace's Equation - On Thursday we will discuss separation of variables for Elliptic Problems, Laplace's Equation: $\nabla^2 u = 0$ where $u = u(x, y)$.

HOMEWORK

1. *Homework Problems*

1. Heat conduction in a thin circular ring of length 2 units allowing for lateral heat losses:

$$
\begin{aligned}
&u_t = \alpha^2 u_{xx} - u \\
&u(-1, t) = u(1, t) \\
&u_x(-1, t) = u_x(1, t) \\
&u(x, 0) = 1 + \cos(\pi x)
\end{aligned}
$$

2. Convection in a thin rod with non-homogeneous boundary conditions. Transform to a PDE with homogeneous boundary condition and without convection. Is your final transformed PDE homogeneous? (Just transform)

$$
\begin{aligned}
&u_t = \alpha^2 u_{xx} - u_x \\
&u(0, t) = 1 \\
&u_{\text{[illegible]}}(1, t) = 0 \\
&u(x, 0) = \phi(x)
\end{aligned}
$$

3. Write a flow chart for how you think about solving PDEs using Separation of Variables and Transforming hard problems into easier ones. Include all of the steps that you take and all of the reminders you might need. Imagine that you are writing yourself an information sheet to be used on an exam.