Day 2 — Partial Differential Eqns.

Farlow: Lessons 2 and 3.

GOAL — gain a better understanding of what our equation means physically so we can test our solutions later.

How to test soln:

A. Check that it matches all boundary and initial conditions

B. PLOT the solution for a few times and see if it makes physical sense.

Diffusion Type Problems

Parabolic Equations — Heat Equation.

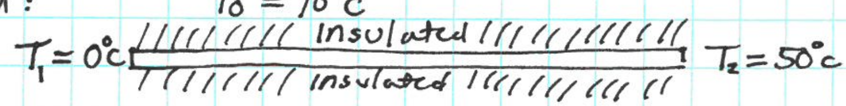
$u_t = \alpha^2 u_{xx}$ The heat Eqn.

"second order, linear, homogeneous, two variables, constant coef"

Why is this called the Heat Eqn?

Heat flow experiment:

$T_0 = 10°c$



What do we expect?

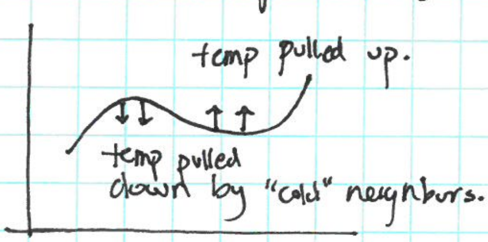
- Temperature equalizes
- settled down to steady state
- linear increasing from right to left.

Heat Equation:

$$u_t = \alpha^2 u_{xx}$$

$$\left[\begin{array}{l}\text{rate of change}\\\text{of temperature}\end{array}\right] = \alpha^2\left[\begin{array}{l}\text{Concavity of the}\\\text{temperature profile}\end{array}\right]$$

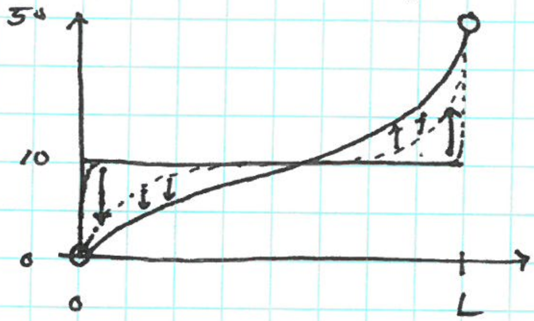
snapshot $u(x, t_n)$



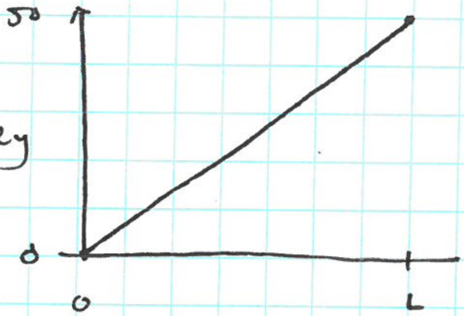
$u_{xx} < 0$ concave down

$u_{xx} > 0$ concave up.

Back to our experiment



eventually



after it becomes a straight line ... no more change!!

FULL PDE for the experiment:

$$u_t = \alpha^2 u_{xx} \qquad 0 < x < L$$

$$u(0,t) = T_1 \qquad \text{Boundary}$$
$$u(L,t) = T_2 \qquad \text{Conditions.}$$

$$u(x,0) = T_0 \qquad 0 \leq x \leq L \qquad \text{Initial Condition.}$$

This is the most basic form of the equation. Now lets play around and add things...

- Heat Gain or Loss along Lateral Boundaries

$$u_t = \alpha^2 u_{xx} - \beta(u - u_0)$$

heat loss / gain is proportional to lab temperature u_0

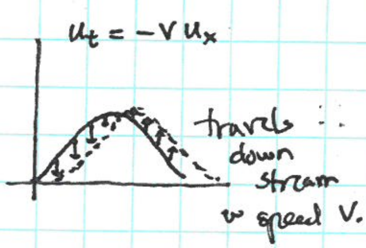
- Internal Heat source

$$u_t = \alpha^2 u_{xx} + f(x,t)$$

Both non-homogeneous

$f(x,t)$ is a heat source or sink

- Diffusion and Convection



$$u_t = \alpha^2 u_{xx} - v u_x$$

strength of convection. magnification

$v u_x$ – how the temp changes because of underlying flow.

Boundary Conditions are Important !!

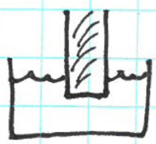
Three main types:

end of rod held a specific temp.

1. $u = g(t)$ "Temperature Forced" (DIRICHLET)

2. $\frac{\partial u}{\partial n} + \lambda u = g(t)$. "Temperature of surrounding media specified" (ROBIN)

Normal derivative across surface



rod in water bath. — bath kept constant.

3. $\frac{\partial u}{\partial n} = g(t)$ "Heat flow across boundary specified" (NEUMAN)

OR (MIXED)

Ex
$$\begin{cases} u_t = \alpha^2 u_{xx} & 0 < x < 200 \quad 0 < t < \infty \\ u_x(0,t) = 0 & \\ u_x(200,t) = -\frac{h}{k}\left(u(200,t) - 20\right) & \\ u(x,0) = 0 & 0 < x < 200 \end{cases}$$

What is this problem telling me?

NO FLUX at $x = 0$ so the $x = 0$ end is insulated.
HEAT FLUX at $x = 200$ amt of flux depends on The difference between the rod and 20°C
INITIALLY The rod is at 0° C.
ONLY DIFFUSION!

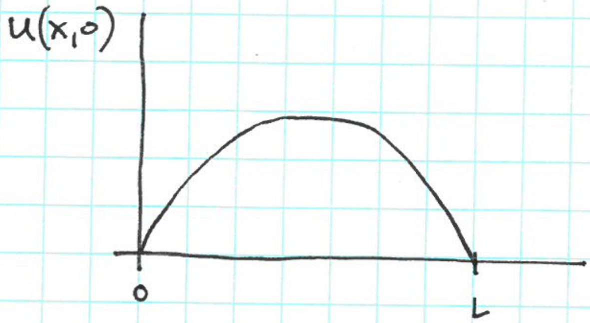
Checking answers:

$$\begin{cases} u_t = \alpha^2 u_{xx} \\ u(0,t) = T_1 \\ u(L,t) = T_2 \\ \\ u(x,0) = T_0 \end{cases}$$

has the soln: $u(x,t) = e^{-(\alpha\pi)^2 t} \sin\left(\frac{\pi x}{L}\right)$

say $T_1 = 0$ $T_2 = 0$

$u(x,0) = \sin\left(\frac{\pi x}{L}\right)$



Check:

IC: when $t=0$ plug into soln

$u(x,0) = e^0 \sin\left(\frac{\pi x}{L}\right) = \sin\left(\frac{\pi x}{L}\right)$ ✓

BCS: when $x=0$

$u(0,t) = e^{-(\alpha\pi)^2 t} \sin(0) = 0$ ✓

$u(L,t) = e^{-(\alpha\pi)^2 t} \sin(\pi) = 0$

Logically what should happen...

Soln decays to $u_\infty(x,t) = 0$, eventually

In Class Questions Day 2

Write down the full IBV (Initial – Boundary Value) Problems for the following systems. Say in words why the equations work and what each term means. Use lambda as heat conduction, alpha squared as the diffusion constant and beta as a constant of proportionality for lateral heat loss.

1. A one dimensional rod of length L with with heat diffusion and lateral heat loss, which is proportional to the difference in temperature between the rod and the ambient temperature of 30 degrees, has one end submerged in a cold bath at 0 degrees and the other end perfectly insulated.
2. A two dimensional square "disk" with sides of length L centered at the origin with a heat source of 10 degrees at the origin is sitting in a lab held at a constant 30 degrees. The edges of the square are in contact with the ambient lab temperature. The disk is assumed thin enough so that there is no need to consider the z-direction and we assume that it is completely insulated in that direction.
3. Make up your own system, fully describe it, and then write down the equations that model that system.